SPATIAL GRAPHS QUESTIONS - LOYOLA 2013

CONFERENCE PARTICIPANTS

Date: June 9, 2013.

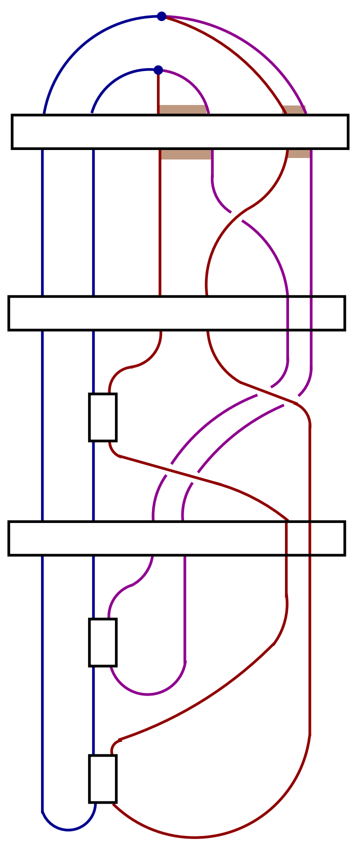
From Scott Taylor

(1) Say that a spatial graph is in *bridge position* if there is an S^2 which separates it into two forests each of which can be isotoped (rel endpoints) into the sphere (which is called a *bridge sphere*). The *bridge number* is half the number of intersection points between the graph and the bridge sphere.

(a) Determine the set of possible bridge numbers for Brunnian θ graphs

(b) Find a normal form for θ graphs of bridge number 5/2 and 3. (Bridge number 5/2 corresponds to what Goda calls "2-bridge". There may already be a normal form for those.)

(c) The figure below depicts a construction (due to Jang, Luitel, Taylor, Zupan) of θ-graphs which are either unknotted or Brunnian. It depends on a pure 4-braid A in the kernel of the map which removes the last two strands. In the top box put the 6-braid obtained by doubling each of the last two strands of A. In the second and third big boxes put the 6-braid obtained from A^{-1} by the usual inclusion map of B_4 into B_6. Put arbitrary numbers of even twists in the other boxes. For a given such braid A, determine the minimal bridge number. (Note the presentation given has bridge number 3)



From Joel Foisy

Final observation: it would be excellent to show directly (that is, without using RST) that if G has a maximal planar subgraph M with a balanced weak conflict graph, then G has a flat embedding.

Conjecture: Give a maximal planar subgraph M, with exactly two $M-$fragments (edges) F and F', if F and F' do not conflict, then there exists a flat embedding of $M \cup F \cup F'$ with F and F' on the same side of $M \subset S^2$ in the flat embedding.

From Mattman Thomas

Questions relating to n-apex graphs

The first three questions are from our arXiv preprint: J. Barsotti and T.W. Mattman, Intrinsically knotted graphs with 21 edges.

Question 1. *Is the Heawood family the set of minor minimal not* 2*–apex (MMN2A) graphs on 21 edges?*

Context: We prove this except for graphs with between 11 and 13 vertices:

Theorem 1. *If* G *is MMN2A with* $|E(G)| = 21$ *and* $|V(G)| \neq 11, 12, 13$*, then* G *is Heawood.*

Question 2. *Does* $Y\Delta$ *preserve N2A on the set of graphs with 21 edges?*

Context: We prove this for graphs of ten or fewer vertices:

Theorem 2. *Suppose* G *has 21 edges and at most 10 vertices. If* G *is N2A and a* $Y\Delta$ *move takes* G *to* G'*, then* G' *is also N2A.*

More From Mattman Thomas

Question 3. *What is the simplest G that is N2A but admits $Y\Delta$ move to G' that's not N2A?*

Context: Assuming the answer to the previous question is yes, we know such a graph has at least 22 edges. Also, $K_{3,3} \sqcup K_{3,3} \sqcup K_{3,3}$ is an example with 18 vertices and 27 edges. The question is asking for a simpler example.

Question 4. *Find a connected (or two component) G that is MMN2A but admits $Y\Delta$ move to G' that's not N2A.*

Context: In other words, a simpler example than $K_{3,3} \sqcup K_{3,3} \sqcup K_{3,3}$

Question 5. *Is there an MMIK graph that is 4–apex?*

Context: It's known that MMIK implies N2A. So an MMIK graph is at least 3-apex. Warning: I haven't even checked the known examples of MMIK graphs yet, so there may be an example there. Since the set of MMIK graphs is finite, we can ask the related question:

Question 6. *What is the smallest n such that every MMIK graph is k–apex for some $k \leq n$.*

Context: As above, we know that $n \geq 3$. It may be that $n = 3$. The first thing to do is check the known examples of MMIK graphs.

I also have some related questions at
http://www.csuchico.edu/math/mattman/PortMattman.pdf

From Hugh Howards

Question Does the complete directed graph on 6 vertices have an oriented link in every embedding?

QUESTIONS

RYO NIKKUNI

(An integral version of) Conway-Gordon type formula was given for each graph G which is obtained from K_7 by a finite sequence of $\triangle Y$-exchanges in [7] ($G = K_7$) and [8] (the others), and for each graph which is obtained from $K_{3,3,1,1}$ by a finite sequence of $\triangle Y$-exchanges in [6].

Question 1. (1) Does a Conway-Gordon type formula hold for $G_{13,30}$ which is a minor-minimal intrinsically knotted graph found in [3]?

(2) Does a Conway-Gordon type formula hold for a minor-minimal intrinsically knotted graph found in [5]?

It is known that for any rectilinear spatial embedding f of K_7, $f(K_7)$ contains a trefoil knot [1], [9], [7]. On the other hand, Hashimoto-Nikkuni showed that for any rectilinear spatial embedding f of $K_{3,3,1,1}$, there exists a Hamiltonian cycle γ of $K_{3,3,1,1}$ such that $a_2(f(\gamma)) > 0$ [6]. Note that all of knots with the stick number less than or equal to 8 and $a_2 > 0$ are 3_1, 5_1, 5_2, 6_3, a square knot, a granny knot, 8_{19} and 8_{20} (Calvo [2]).

Question 2. For any rectilinear spatial embedding f of $K_{3,3,1,1}$, does there exist a Hamiltonian cycle γ of $K_{3,3,1,1}$ such that $f(\gamma)$ is a trefoil knot?

The following is also still open, as far as the author know.

Question 3. (Foisy-Ludwig [4]) For any spatial embedding f of $K_{3,3,1,1}$ (which does not need to be rectilinear), does there exist a Hamiltonian cycle γ of $K_{3,3,1,1}$ such that $f(\gamma)$ is a nontrivial knot?

References

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