Topological Symmetry Groups of the Heawood Graph

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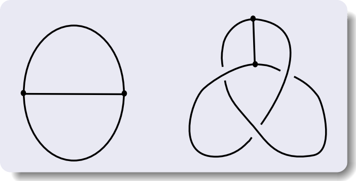
November 9, 2019

Introduction

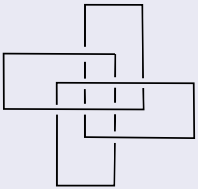
- The study of spatial graphs is as a generalization of the study of knots and links.



- The study of knot symmetries has a long history in knot theory.

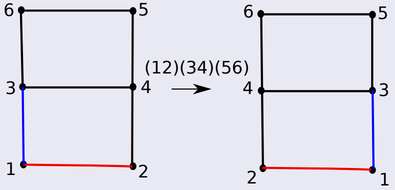
Figure 8

Figure 8



Definition

An **automorphism** of a graph γ is a permutation σ of the vertices if a pair of vertices u and v are adjacent then $(\sigma(u), \sigma(v))$ are adjacent. $Aut(\gamma)$ is the group of automorphisms of γ.

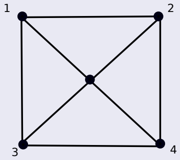


We may identify $Aut(\gamma)$ with a subgroup of S_n, where n is the number of vertices in γ.

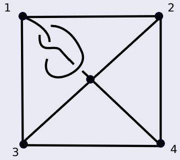
Introduction

- Consider the relationship between the automorphisms of an abstract graph and homeomorphisms of particular embeddings of the graph in S^3.

(1234) induced by rotation



(1234) not induced by any homeo.



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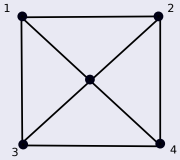

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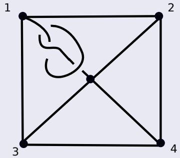

- Some embeddings of the abstract graph in S^3 have homeomorphisms that induce a given automorphism and others don't.

Realizable Automorphisms

Question: Given a single automorphism σ of an abstract graph γ, can σ be induced by a homeomorphism of some embedding of γ in S^3?

Definition

An automorphism σ of an abstract graph γ is **realizable** (resp. **positively realizable**) if there is some embedding Γ of γ in S^3 such that the automorphism is induced by a homeomorphism (resp. orientation preserving homeo.) of (S^3, Γ).

Realizable Automorphisms

Given K_5 and $(1234) \in Aut(K_5)$, is there an embedding of K_5 that realizes this automorphism?

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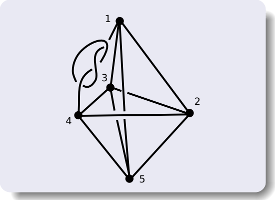
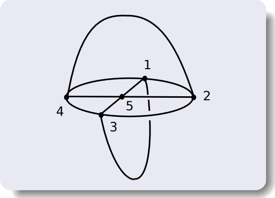




- (1234) can't be induced by any homeomorphism of the embedding Γ on the left.
- (1234) can be induced by a homeomorphism of the embedding Γ' on the right since there is a homeomorphism of (Γ', S^3) that rotates the picture clockwise by $\frac{\pi}{2}$ then reflects.
- Thus, (1234) is a realizable automorphism of K_5.

Realizable Automorphisms

For $n \leq 5$, every automorphism in $Aut(K_n) = S_n$ can be induced by some homeomorphism of some embedding of the graph in S^3.

But what about K_6?

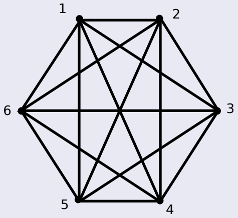
Realizable Automorphisms

For $n \leq 5$, every automorphism in $Aut(K_n) = S_n$ can be induced by some homeomorphism of some embedding of the graph in S^3.

But what about K_6?

Question: Given automorphism $(1234) \in Aut(K_6)$, is there an embedding that realizes this automorphism?

K_6



Realizable Automorphisms

Theorem (Flapan, 1989)

For any embedding of K_6 in S^3, and any labelling of the vertices of K_6 by the numbers 1 *through* 6*, there is no homeomorphism $h : (S^3, K_6) \to (S^3, K_6)$ which realizes the automorphism* (1234) *on the vertices of K_6.*

Topological Symmetry Groups

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Definition

The **topological symmetry group** $TSG(\Gamma)$ of a graph Γ embedded in S^3, is the subgroup of $Aut(\Gamma)$ induced by homeomorphisms of (S^3, Γ).

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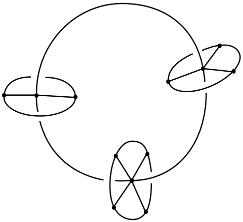
The **orientation preserving topological symmetry group** $TSG_+(G)$ is the subgroup of $Aut(\Gamma)$ induced by orientation preserving homeomorphisms of (S^3, Γ).

Note that either $TSG_+(\Gamma)$ is an index two subgroup of $TSG(\Gamma)$ or $TSG_+(\Gamma) = TSG(\Gamma)$.

Definition

Let G be a group and let γ denote an abstract graph. If there is some embedding Γ of γ in S^3 such that $TSG(\Gamma) = G$ (resp. $TSG_+(\Gamma) = G$) then the group G is said to be **realizable** (resp. **positively realizable**) for γ.



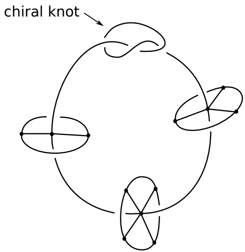
$$TSG(\Gamma) \cong (\mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_4) \rtimes \mathbb{Z}_2$$

However, restricting to orientation preserving homeomorphisms gives:

$$TSG_+(\Gamma) \cong \mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_4$$

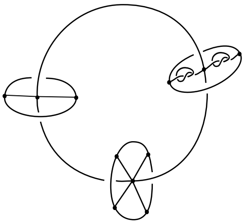
Topological Symmetry Groups

Since the knot is chiral it's not isotopic to its mirror image. Hence there is no orientation reversing homeomorphism of (S^3, Γ).



$$TSG(\Gamma) = TSG_+(\Gamma) \cong \mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_4$$

Topological Symmetry Groups

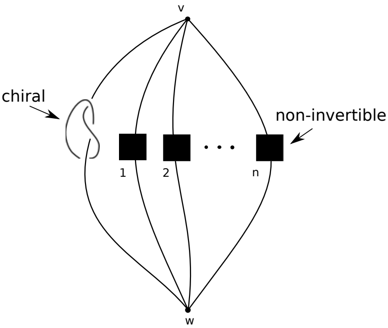


$$TSG(\Gamma) \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_4$$

If two wheels have the same number of spokes then add a chiral knot to each spoke of one of the wheels.

Any finite abelian group can be $TSG(\Gamma)$.

Topological Symmetry Groups



Any transposition (ij) is induced by twisting strands.

Non-invertible knots $\Rightarrow$ no homeomorphism interchanges v and w.

Thus $TSG(\Gamma) = S_n$

Topological Symmetry Groups

Question: Is every finite group isomorphic to $TSG(\Gamma)$ for some graph Γ embedded in S^3?

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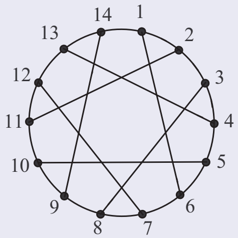
Theorem (Flapan, Naimi, Pommersheim, Tamvakis 2005)

$TSG(\Gamma)$ can be A_n for some graph Γ embedded in S^3 if and only if $n \leq 5$.

The Heawood graph

The Heawood graph C_{14}



- The Heawood graph is obtained from K_7 by $\Delta - Y$ moves and hence is intrinsically knotted. (Nikkuni and Taniyama, 2012)
- C_{14} is intrinsically chiral, hence no embedding of C_{14} in S^3 has an orientation reversing homeomorphism thus $TSG(\Gamma) = TSG_+(\Gamma)$ for each embedding Γ of C_{14}.(Flapan, Fletcher, Nikkuni, 2014)

More Facts about C_{14}:

- $Aut(C_{14}) = PGL(2,7)$ which has order $336 = 2^4 \times 3 \times 7$. (Coxeter, 1950)
- $Aut(C_{14}) = PGL(2,7)$ has non-trivial elements of order 2, 3, 4, 6, 7, and 8. (Magma)

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- $Aut(C_{14}) = PGL(2,7)$ has non-trivial elements of order 2, 3, 4, 6, 7, and 8. (Magma)

Question: Which of these orders of elements are possibilities for realizable automorphisms of C_{14}?

Classification of realizable automorphisms of C_{14}.

Classification of realizable automorphisms of C_{14}

Theorem (Lawrence, Flapan, W.)

C_{14} has no realizable automorphism of order 4 or 8.

To rule out orders 4 and 8 we rely on a Conway-Gordon type theorem for C_{14} (Nikunni):

- For any embedding of C_{14} in S^3, the mod 2 sum of the arf invariants of all 12- and 14-cycles is 1.

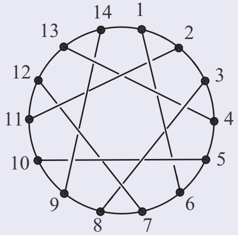
Classification of realizable automorphisms of C_{14}

Theorem (Lawrence, Flapan, W.)

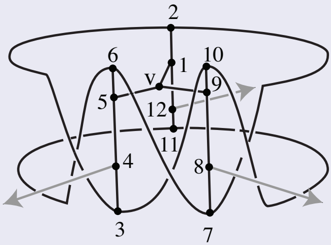
A non-trivial automorphism of C_{14} is realizable if and only if it has order 2, 3, 6, *or* 7.

Classification of realizable automorphisms of C_{14}

Orders 7 is realizable



Order 6 is realizable



- In the embedding on the left a rotation of order 7 realizes the automorphism $(1, 3, 5, 7, 9, 11, 13)(2, 4, 6, 8, 10, 12, 14)$.
- In the embedding on the right we can see that $(v, w)(10, 11, 6, 7, 2, 3)(1, 4, 9, 12, 5, 8)$ has order 6 and is induced by a glide rotation h.
- h^2 and h^3 induce automorphisms of order 3 and 2 respectively.

Classification of Topological Symmetry Groups of of C_{14}

Topological Symmetry Groups of C_{14}

Since C_{14} is chiral, for any embedding Γ of C_{14} it follows that:

- $TSG_+(\Gamma) = TSG(\Gamma)$.
- $TSG(\Gamma)$ is a subgroup of $Aut(C_{14}) = PGL(2,7)$

The non-trivial subgroups of $PGL(2,7)$ are:

$\mathbb{Z}_2$, $\mathbb{Z}_3$, $\mathbb{Z}_4$, $\mathbb{Z}_6$, $\mathbb{Z}_7$, $\mathbb{Z}_8$, D_2, D_3, D_4, D_6, D_7, D_8, A_4, S_4, $\mathrm{PSL}(2,7)$, $\mathbb{Z}_7 \rtimes \mathbb{Z}_3$, and $\mathbb{Z}_7 \rtimes \mathbb{Z}_6$.

Topological Symmetry Groups of C_{14}

C_{14} has no realizable automorphism of order 4.

This rules out the following groups as possibilities for $TSG(\Gamma)$:

$$\mathbb{Z}_2, \mathbb{Z}_3, \mathbb{Z}_4, \mathbb{Z}_6, \mathbb{Z}_7, \mathbb{Z}_8, D_2, D_3, D_4, D_6, D_7, D_8, A_4, S_4, \mathrm{PSL}(2,7), \mathbb{Z}_7 \rtimes \mathbb{Z}_3, \text{ and } \mathbb{Z}_7 \rtimes \mathbb{Z}_6.$$

Remaining possibilities for $\mathrm{TSG}(\Gamma)$ for some embedding Γ of C_{14} are:

$$\mathbb{Z}_2, \mathbb{Z}_3, \mathbb{Z}_6, \mathbb{Z}_7, D_2, D_3, D_6, D_7, A_4, \mathbb{Z}_7 \rtimes \mathbb{Z}_3, \text{ and } \mathbb{Z}_7 \rtimes \mathbb{Z}_6.$$

Topological Symmetry Groups of C_{14}

Theorem (Lawrence, Flapan, W.)

The groups D_2, D_6, A_4, $\mathbb{Z}_7 \rtimes \mathbb{Z}_3$, and $\mathbb{Z}_7 \rtimes \mathbb{Z}_6$ are not realizable for C_{14}.

Topological Symmetry Groups of C_{14}

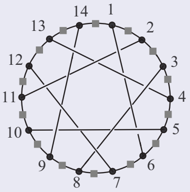
Theorem (Lawrence, Flapan, W.)

The trivial group and the groups $\mathbb{Z}_2$, $\mathbb{Z}_3$, $\mathbb{Z}_6$, $\mathbb{Z}_7$, D_3, and D_7 are realizable for C_{14}.

Topological Symmetry Groups of C_{14}

In this embedding Γ of C_{14} the grey squares in the outer circle C represent the same trefoil knot. Any homeomorphism of C_{14} in S^3. must take C to itself.

$TSG(\Gamma) = D_7$



- Γ is invariant under a $\frac{2\pi}{7}$ rotation inducing an order 7 automorphism.
- Turning C over induces on order 2 automorphism.
- $TSG(\Gamma) = D_7$.

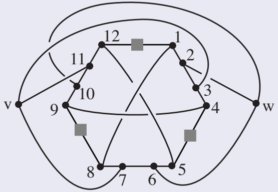
Theorem (Subgroup Theorem)

Let Γ be a 3-connected graph embedded in S^3 which has an edge e that is not pointwise fixed by any non-trivial element of $TSG_+(\Gamma)$. Then for every (possibly trivial) subgroup H of $TSG_+(\Gamma)$ there is an embedding Γ' of Γ with $H = TSG_+(\Gamma')$.

By the Subgroup Theorem, since $TSG_+(\Gamma) = D_7$ it follows that $\mathbb{Z}_7$, $\mathbb{Z}_2$, and the trivial group are also realizable.

In this embedding Γ' of C_{14} the grey squares in the outer hexagon C represent trefoil knots so any homeomorphism must take C to itself.

$TSG(\Gamma') = D_3$

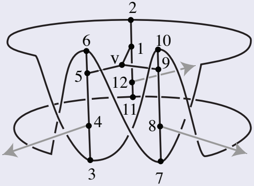


- Γ' is invariant under a $\frac{2\pi}{3}$ rotation inducing an order 3 automorphism.
- Turning C over induces an order 2 automorphism.
- $TSG(C_{14}) = D_3$.
- Replacing three trefoils with non-invertible knots gives embedding Γ'' with $TSG(\Gamma'') = \mathbb{Z}_3$.

Topological Symmetry Groups of C_{14}

Let Λ be the embedding below.

$TSG(\Lambda) = \mathbb{Z}_6$



- The 6-cycle C is the only one that contains a trefoil knot so it is setwise invariant under any homeomorphism of Λ in S^3.
- A glide rotation induces an automorphism of order 6.
- D_6 is not realizable.
- $TSG(\Lambda) = \mathbb{Z}_6$.

Thanks!

THANK YOU

Special thanks to:

- Alissa, Blake, and Pat
- UCR and AMS
- Erica and Emille

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