Genus 2 Goeritz Equivalence in S^3

AMS Sectional Meeting

Special Session on Invariants of Knots and Spatial Graphs

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(Joint with Brandy Doleshal)

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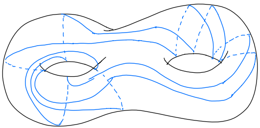
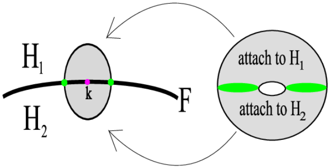
Motivation

Eudave-Muñoz–Miyazaki–Motegi and Doleshal find infinite examples of primitive/Seifert knots that have inequivalent positions (but the same surface slope) on a genus 2 Heegaard splitting of S^3.



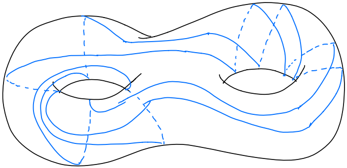


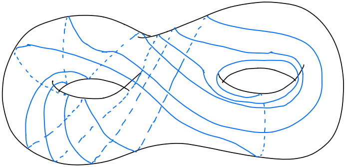
Source: Michael Williams

Both show they are distinct positions by using the orders of the singular fibers in the Seifert-fibered space obtained by a handle-attachment to one of the handlebodies.

Motivation

Is there an easier way to determine whether two curves on a genus two Heegaard surface in S^3 are equivalent?

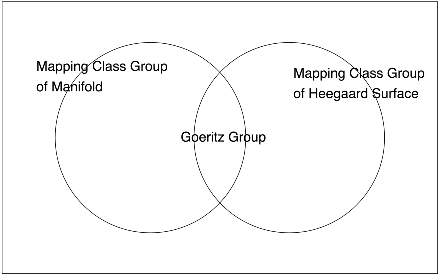




Goeritz Group

Intuition

The **Goeritz group** of a Heegaard splitting (F, H_1, H_2) of a 3-manifold M is the group of isotopy classes of orientation-preserving diffeomorphisms of M that preserve (F, H_1, H_2).



We say two curves on a genus two Heegaard surface are **equivalent** if they are related by a(n) (extended) Goeritz group element.

Goeritz Group

Presentation

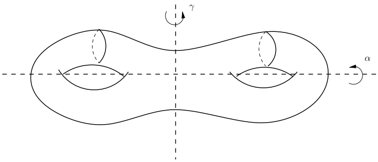
From Scharlemann and Akbas, for S^3, and $g = 2$:

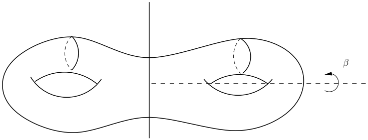
$$\mathcal{G} = \langle \alpha \mid \alpha^2 \rangle \oplus \langle \beta, \gamma, \delta \mid \gamma^2,\ (\beta^{-1}\gamma\delta)^3,\ \alpha\gamma\beta\gamma\beta^{-1},\ (\beta^{-1}\delta)^2 \rangle,$$

where,

δ is a handle slide,

and α, β, and γ are as shown:





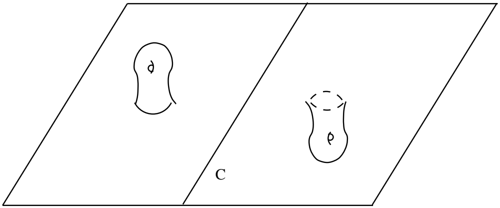
Extended Goeritz Group

$$\widehat{\mathcal{G}} = \mathcal{G} \rtimes \langle \varepsilon \mid \varepsilon^2 \rangle,$$

$$\widehat{\mathcal{G}} = \langle \alpha \mid \alpha^2 \rangle \oplus \langle \beta, \gamma, \delta, \varepsilon \mid \gamma^2,\ (\beta^{-1}\gamma\delta)^3,\ \alpha\gamma\beta\gamma\beta^{-1},\ (\beta^{-1}\delta)^2, \alpha\varepsilon\beta\varepsilon\beta^{-1}, \varepsilon\gamma\varepsilon\gamma, \varepsilon\delta\varepsilon\delta \rangle.$$

where,

ε exchanges the two handlebodies.



Action on Homology

Homomorphism

$$* : \mathcal{G} \to GL(4, \mathbb{Z})$$

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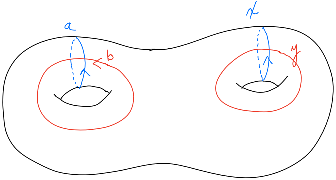
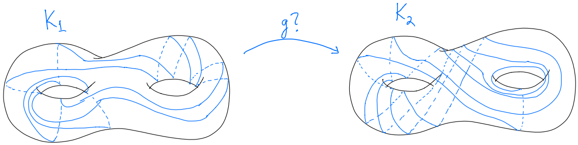


$$\begin{aligned}
\alpha_*((a, x, b, y)) &= (-a, -x, -b, -y) \\
\beta_*((a, x, b, y)) &= (a, -x, b, -y) \\
\gamma_*((a, x, b, y)) &= (-x, -a, -y, -b) \\
\delta_*((a, x, b, y)) &= (a + x, x, b, y - b) \\
\varepsilon_*((a, x, b, y)) &= (y, b, x, a)
\end{aligned}$$

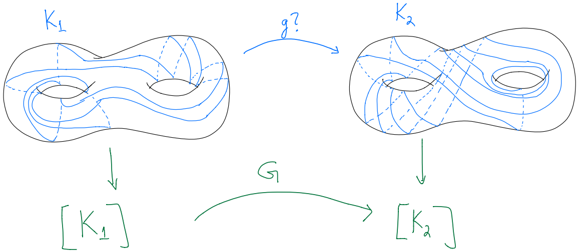
Action on Homology

Big Idea



Action on Homology

Big Idea



Action on Homology

Group Theory

Image of $* : \mathcal{G} \to GL(4, \mathbb{Z})$ is the subgroup

$$\mathcal{D} = \left\{ \begin{pmatrix} C & 0 \\ 0 & (C^{-1})^T \end{pmatrix} \mid C \in GL(2, \mathbb{Z}) \right\} \cong GL(2, \mathbb{Z}),$$

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Split Product

The actions of the (non-extended) Goeritz group on homology act separately (though not independently) on the first two and the last two factors of the homology vector.

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Definition

The **split product** of the homology vector (a, x, b, y) is

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Theorem

If K is a simple closed curve on a genus 2 Heegaard surface in S^3, and $[K] = (a, x, b, y)$ is the homology vector of K with respect to the standard basis, then $SP([K])$ is equal to the surface slope.

Algorithm

Theorem

Suppose K_1 and K_2 are simple closed curves in the genus two Heegaard surface for S^3 with non-zero surface slope. There is an algorithm to determine whether K_1 and K_2 are Goeritz equivalent.

Algorithm Outline

Given K_1 and K_2 on Heegaard surface F.

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1. Find $[K_1] = (a, x, b, y)$, $[K_2] = (a', x', b', y')$, $m = ab + xy$, $m' = a'b' + x'y'$, $d = gcd(a, x)$, $d' = gcd(a', x')$. If $m \neq m'$ or $d \neq d'$, $K_1 \ncong K_2$.

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   If $m \neq m'$ or $d \neq d'$, $K_1 \ncong K_2$.
2. $$\begin{pmatrix} A & 0 \\ 0 & (A^T)^{-1} \end{pmatrix} \begin{pmatrix} a \\ x \\ b \\ y \end{pmatrix} = \begin{pmatrix} a' \\ x' \\ b' \\ y' \end{pmatrix} \iff \begin{array}{ccc} A(a, x)^T & = & (a', x')^T \\ (A^T)^{-1}(b, y)^T & = & (b', y')^T \end{array}$$

   Linear algebra $\Rightarrow$ At most one matrix A

Algorithm Outline

3. Factorize $G = \begin{pmatrix} A & 0 \\ 0 & (A^T)^{-1} \end{pmatrix}$ into generators $\alpha_*, \beta_*, \delta_*, \gamma_*$.
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4. Check whether $\beta^{2n} g(K_1) = K_2$.
5. (Repeat for K_1 and $\varepsilon(K_2)$).

Split Orthogonality

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Let $K_1 = K_2$ with $[K_1] = [K_2] = (1, 0, 0, 0)$.
Then for all $n \in \mathbb{Z}$,

$$\beta(\gamma\delta\gamma)^n$$

is non-trivial, and

$$\beta_*(\gamma_*\delta_*\gamma_*)^n$$

fixes $[K_1] = (1, 0, 0, 0)$.

Thank you all for your attention!

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