n-Flat Graphs

Ramin Naimi

Joint work with Hedda Wanchen Zhao
(Preliminary report)

Most of this work was originally done in 2003 by Ryo Nikkuni (Tokyo Woman's Christian University) and Yukihiro Tsutsumi.

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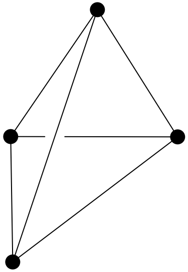
Example. A paneled embedding of K_4:



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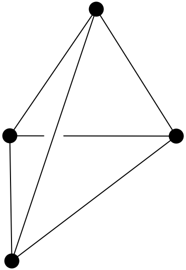
Example: K_4 is flat because it has a paneled embedding:



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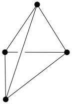
Example: K_4 is 2-flat (by inspection). 

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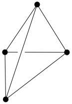

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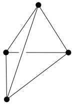

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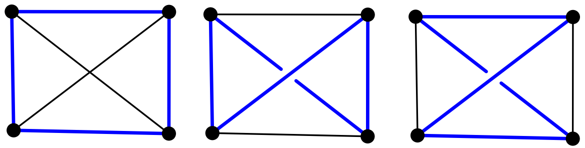

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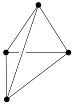

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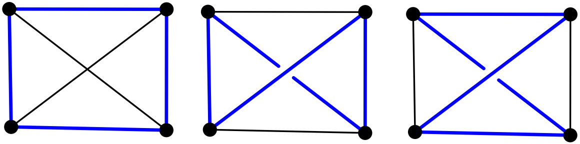

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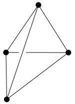

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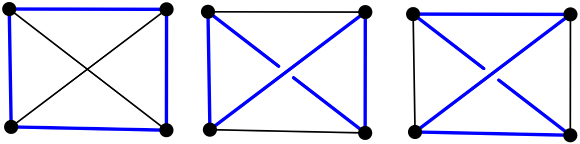

Take any embedding, and any panels for these three cycles. Glue together abstractly, they make a projective plane.
So the three panels cannot be embedded in $\mathbb{R}^3$ with pairwise disjoint interiors.

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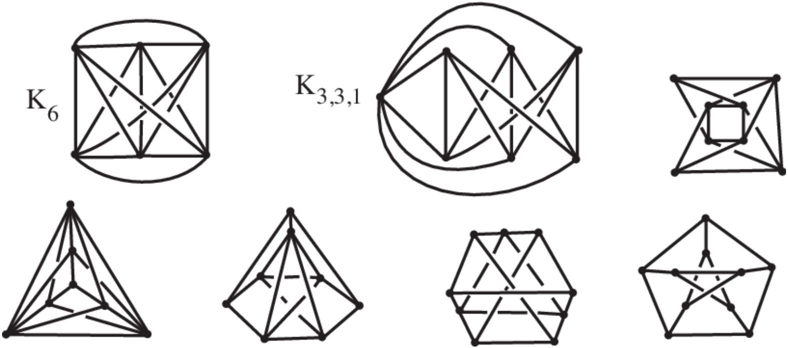
So Γ is n-flat iff it has no minor in Ω_n.
Ω_n is the **obstruction set** for n-flatness.

Ω_1

Theorem (Robertson, Seymour, Thomas 1985)

Ω_1 := *the Petersen family of graphs.*



Ω_n

Graph Minor Theorem (Robertson & Seymour) $\implies$ Ω_n is finite.

Ω_n

Theorem

(Almost)

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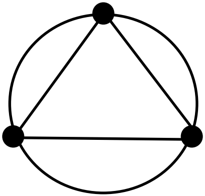

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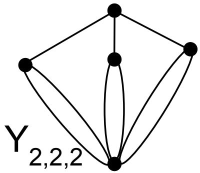

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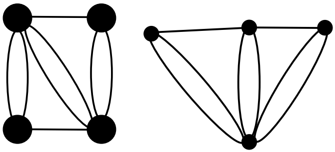

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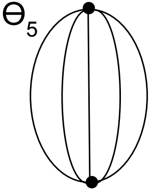

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- $\Omega_2 \supseteq \{K_6, K_{3,4}\}$ *(conjectured equality)*

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If the conjecture is true, the answer is: 2-flatness.

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Recall: K_4 is not 3-flat.
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Now can show Γ is 3-flat...

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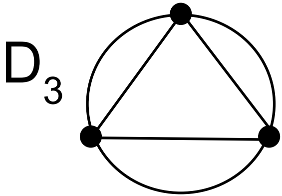
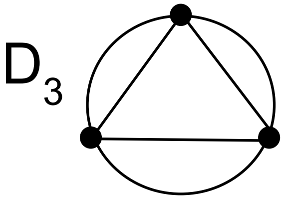


D_3 is not 4-flat.

D_3



D_3 is not 4-flat.

Proof: It has four 3-cycles whose panels, when glued together, yield a projective plane.

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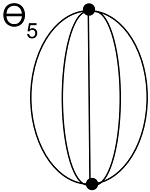
Proof: Take an arbitrary embedding of θ_5 and a panel for each of the ten cycles.



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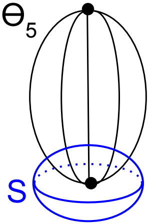

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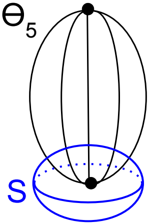

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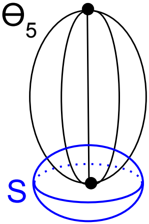

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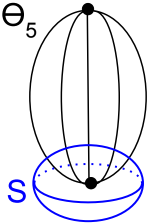

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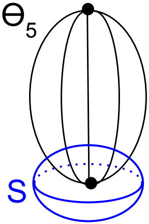

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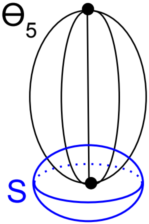

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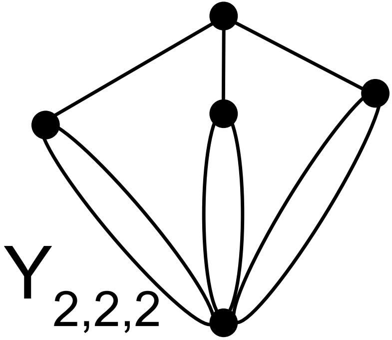
$\implies$ At least two panels intersect in their interiors.

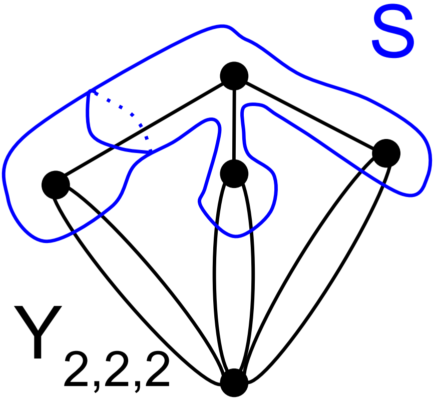
$Y_{2,2,2}$

$Y_{2,2,2}$ is not 6-flat.



$Y_{2,2,2}$

$Y_{2,2,2}$ is not 6-flat.



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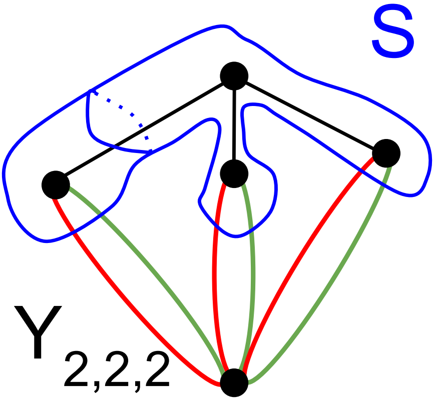
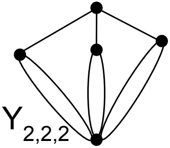
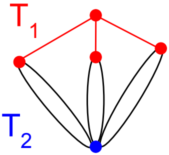


Lemma for proving the main theorem

Lemma

If Γ has no K_4 minor and no D_3 minor, then Γ contains disjoint trees T_1, T_2 such that $V(T_1) \cup V(T_2) = V(\Gamma)$ and for every cycle C in Γ, $C \cap T_i$ is connected.





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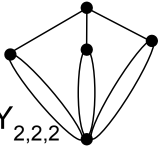
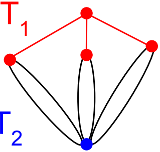
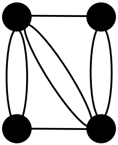
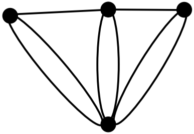


$Y_{2,2,2}$



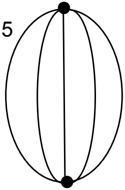


$N_{2,2,2}$



$V_{2,2,2}$

Θ_5



Thank you!