Most graphs are knotted

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Abstract

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Abstract: We present four models for a random graph and show that, in each case, the probability that a graph is intrinsically knotted goes to one as the number of vertices increases. We also argue that, for $n \geq 18$, most graphs of order n are intrinsically knotted and, for $n \geq 2m + 9$, most of order n are not m-apex.

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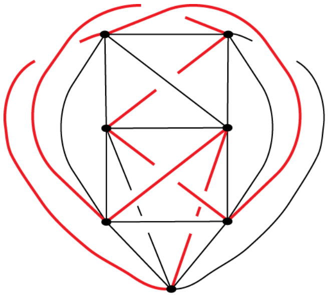
Theorem -2 (Conway & Gordon, 1983)

K_7 is intrinsically knotted.



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Here are four models.

Four models of a Random Graph

Let $N = \binom{n}{2}$ denote the number of edges in K_n.
1. (Erdös-Rényi, 1959) Choose a graph $G(n, M)$ uniformly at random from the set of labelled graphs with $|V| = n$ and $|E| = M$.

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3. (Unlabelled version of 2.5) Let Γ_n denote the number of unlabelled graphs on n vertices. Choose a graph from this set uniformly at random.

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A1. In model 2.5 and 3, there's a constant n_{IK} such that, when $n \geq n_{IK}$, most (at least half) graphs with n vertices are IK.

A2. In all four models, the probability that a graph is IK goes to one as the number of vertices increases.

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Every subgraph is a minor, but minor is a bigger class.
Think of a minor as a "topological" subgraph.

Key Observation

Theorem -1

If $|V| = n \geq 7$ *and* $|E| \geq 5n - 14$*, then* G *is intrinsically knotted.*

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Follows from Mader (1968): such a graph has a K_7 minor.

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Lemma 2

When $n \geq 18$, either G or its complement is intrinsically knotted.

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G is 2-apex if there are vertices a and b so that $G - a, b$ is planar.

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Since $n \geq 18$, we have $|E| \geq 5n - 14$.

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It follows that at least half the graphs are intrinsically knotted.

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The last inequality is due to Hoeffding, with $t = p - (5n - 15)/N$ and shows probability goes to 0 as n goes to infinity. □

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Similarly define n_{NmA} for the "not m-apex" property.

A graph is m-apex if there are m (or fewer) vertices whose deletion makes G planar.

m-apex graphs

Theorem 3

In Model 2.5 and 3, if $n \geq n_{NmA}$, then at least half the graphs are not m-apex.

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For example, there is a self-complementary planar G with $|V| = 8$.
For $m > 0$, the construction is due to Pavelescu & Pavelescu, 2017.

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Deleting at most $2m$ vertices, create subgraph H with $|V_H| \geq 9$ and H and its complement both planar.

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Deleting at most $2m$ vertices, create subgraph H with $|V_H| \geq 9$ and H and its complement both planar.

This contradicts BHK. □