Intrinsic Linking in Directed Graphs

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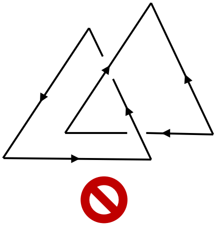
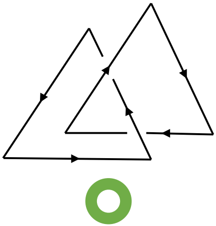
Definitions

Definition

A graph G is called *intrinsically linked* if every embedding of G into 3-space contains disjoint cycles that form a non-split link L. A directed graph G is called *intrinsically linked as a directed graph* if every embedding of G into 3-space contains disjoint cycles that form a non-split link L, and further, the edges of G that make up each component of L have a consistent orientation.





Intrinsically linked directed graphs exist

A directed graph may have up to 2 edges between each pair of vertices: one from v to w and one from w to v.

Definition

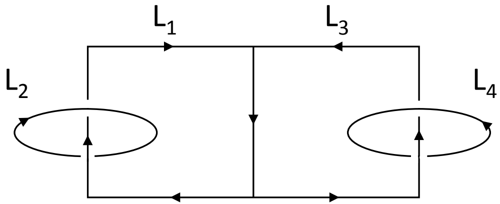
A directed graph with all possible edges is called a *symmetric complete directed graph*.

Theorem (Foisy-Howards-Rich)

The symmetric complete directed graph on 6 vertices is intrinsically linked as a directed graph.

The Gluing Lemma Fails

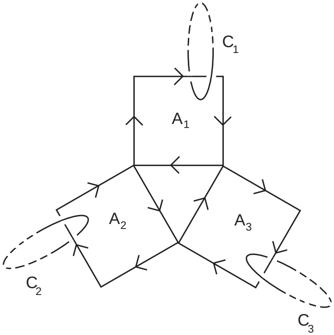
The restriction that the components of L inherit a consistent orientation from G is restrictive and causes many differences between the directed and classical cases.



The standard Gluing Lemma fails for directed graphs: a 3-link exists, but it is not consistently oriented.

Intrinsic n-linking

One way to build intrinsically n-linked graphs for $n > 2$ is to use the pigeonhole principle.



This approach has worked for small n.

Intrinsic n-linking

We can also find *n*-linked directed graphs by adapting techniques of Flapan-Mellor-Naimi.

Theorem

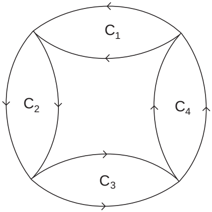
There exist intrinsically n-linked directed graphs for all n.

Theorem (Mattman-Naimi-Pagano)

Intrinsic linking and knotting are arbitrarily complex in directed graphs (in the sense of Flapan-Mellor-Naimi).

Intrinsic Knotting

To construct an intrinsically knotted graph, it suffices to find a D_4 minor with certain linking conditions [Taniyama-Yasuhara and Foisy]. This can be adapted to directed graphs.

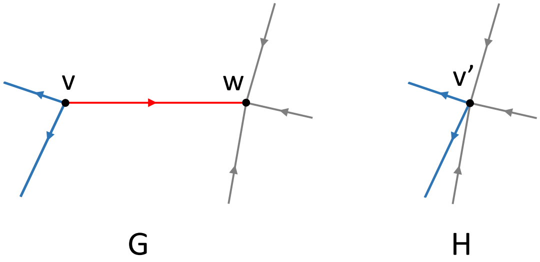


However, we cannot rely on the standard minor operation in the directed graph case.

Minors are not well behaved

In the classical case, if H is minor of G and G has a linkless embedding, so does H.

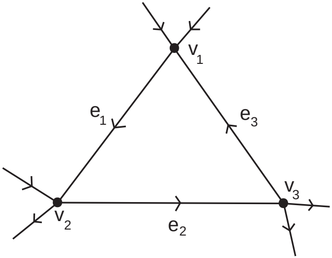
This does not hold in the case of directed graphs, as the number of consistently oriented cycles may increase when moving from G to H.



A well behaved minor-like operation

Definition

Let e be an edge directed from v to w in G. If v is a sink in $G \setminus e$ **or** w is source in $G \setminus e$, then if H is obtained from G by contracting e, we say H is obtained from G by a *consistent edge contraction*.



Well behaved minor-like operations

Proposition

If G has a linkless embedding and H is obtained from G by a consistent edge contraction, then H has a linkless embedding as well.

A second operation called *H-cyclic subcontraction* also preserves linkless embeddings under the assumption that all intrinsically linked directed graphs contain at least one link with non-zero linking number.

Density and intrinsic linking

Definition

Let $E(G)$ be the number of edges in a graph G. Then the *density* of a graph G on n vertices is $\frac{E(G)}{E(K_n)}$.

There is a constant c such that if G has $n > 6$ vertices and more than cn edges, then G has a K_6 minor [Mader].

Corollary

Let G be an undirected graph on $n > 6$ vertices. There exists $\delta_{IL}(n)$ such that if the density of $G > \delta_{IL}(n)$ then G is intrinsically linked. Further, $\delta_{IL}(n) \to 0$ as $n \to \infty$.

Density and intrinsic linking in directed graphs

Definition

For a directed graph G on n vertices, the density of G is $\frac{E(G)}{2E(K_n)}$.

For all n, we are able to construct a directed graph G on n vertices that is not intrinsically linked as a directed graph, with the density of $G > \frac{1}{2}$.

Proposition

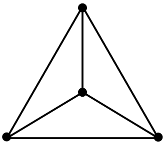
For all $n > 6$, there exists a density $\delta_{IL}(n)$ such that if G is a directed graph on n vertices and the density of $G > \delta_{IL}(n)$, then G is intrinsically linked as a directed graph. Further, $\frac{1}{2} < \delta_{IL}(n) \leq \frac{9}{10}$ for all n.

Tournaments

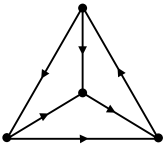
Definition

A *tournament* is a directed graph with exactly one edge between each pair of vertices.

Equivalently, a tournament is a complete graph together with a choice of orientation for each edge.



K_4



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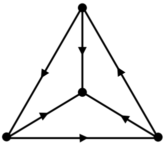

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Linking in Tournaments

Moving from complete graphs to tournaments meaningfully changes which graphs are intrinsically linked or knotted.

Theorem

K_6 is intrinsically linked [Conway-Gordon, Sachs] and K_7 is intrinsically knotted [Conway-Gordon].

Proposition

The smallest intrinsically linked tournament has 8 vertices.

Proposition

No tournament on 8 or fewer vertices is intrinsically knotted, and there exists a tournament on 12 vertices that is intrinsically knotted as a directed graph.

The consistency gap

Definition

The *consistency gap* is $m' - m$ where K_m is the smallest intrinsically $n-$linked complete graph, and m' is the number of vertices in the smallest tournament that is intrinsically $n-$linked as a directed graph. We denote it $cg(n)$.

$$cg(2) = 2$$

$$0 \leq cg(3) \leq 13$$

$$0 \leq cg(4) \leq 54$$

$$0 \leq cg(5) \leq 139$$

$$0 \leq cg(n) \leq \lceil 7.5(2n-3)^2 \rceil - 3n$$

Questions

- What is the right analogue of minors for intrinsic linking in directed graphs? Consistent edge contraction and H-cyclic subcontraction each have drawbacks.
- What is the smallest number of vertices in an intrinsically knotted directed graph ($7 \leq n \leq 11$) or tournament ($9 \leq n \leq 12$)?
- What is the growth rate of the consistency gap? $O(n^2)$? $O(n)$? Is it bounded?

References

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T. Fleming, J. Foisy, *Intrinsically knotted and 4-linked directed graphs*, J. Knot Theory Ramifications 27 (2018) no.6, 1850037-1 to 1850037-18

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