Moves for isotopic singular link cobordisms in 4-space

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This talk is based on the following paper:

Carmen Caprau, *Movie moves for singular link cobordisms in 4-dimensional space*, J. of Knot Theory and its Ramifications 25, No. 2 (2016); arXiv:1507.04077 [math.GT].

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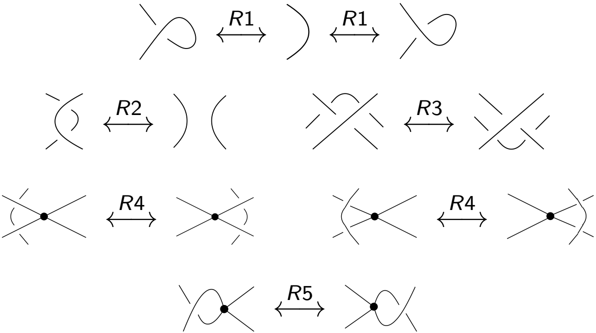
Singular knots

- A singular knot is a rigid-vertex embedding of a 4-valent graph in $\mathbb{R}^3$.
- Equivalently, a singular knot is an *immersion* of a disjoint union of circles in $\mathbb{R}^3$, which has finitely many singularities that are all transverse double points.

- A diagram of a singular knot is a generic projection of the knot into a plane.

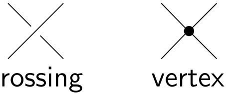
- Two diagrams represent isotopic singular knots if they differ by a finite sequence of the following moves:



- We refer to these moves for diagrams as the extended Reidemeister moves.

Analyzing the projection of an isotopy (of singular knots)

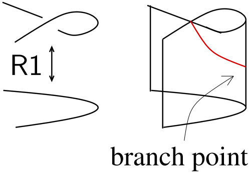
- Diagrams for singular knots contain 0-dim'l singular sets:

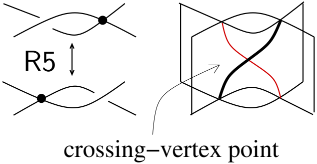


- As an isotopy of singular knots occurs, these 0-dim'l singular sets trace 1-dim'l sets.
- The philosophy of the extended Reidemeister moves is to study the <u>transverse intersections</u> and <u>critical points</u> of the resulting 1-dim'l sets.

- **Observation:** The moves for singular knot diagrams do <u>not</u> correspond to critical points for the vertex set of a diagram (since during an isotopy of singular knots the topology of the underlying graph remains unchanged.)

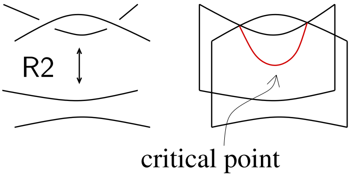
Critical points – type I





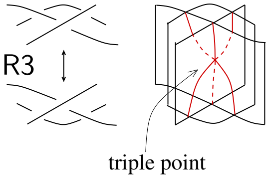
- The moves $R1$ and $R5$ correspond to a critical point of the double-point set.
- A move $R1$ corresponds to a branch point.
- A move $R5$ corresponds an isolated singular point, which we call a crossing-vertex point, where an edge (vertex $\times[0,1]$) intersects the arc of a double-point set (crossing $\times[0,1]$).

Critical points – type II



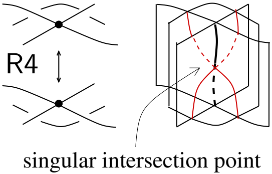
- The move $R2$ corresponds to a critical point (maximum or minimum) of the double-point set.

Transverse intersections – type III



- The move $R3$ corresponds to an isolated triple point.
- During a move $R3$, a crossing point traces a 1-dim'l set (crossing point $\times[0, 1]$) and intersects transversally a 2-dim'l sheet (arc $\times[0, 1]$).

Transverse intersections – type III

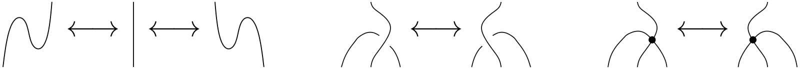


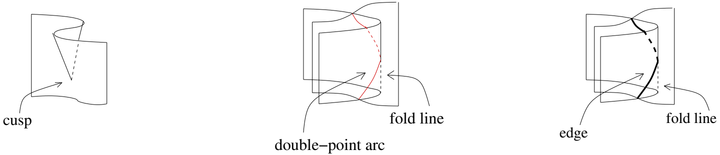
- The move $R4$ corresponds to a transverse intersection between a 1-dim'l set (vertex $\times[0,1]$) and a 2-dim'l sheet (arc $\times[0,1]$).
- This move creates an isolated point which we refer to as a singular intersection point.

Remark: The moves $R1$ through $R5$ exhaust the possibilities for critical points and transverse intersections in $\mathbb{R}^3$ as an isotopy between singular knot diagrams occurs. Therefore, these five moves are sufficient to relate isotopic singular knot diagrams.

Additional extended Reidemeister moves with height functions

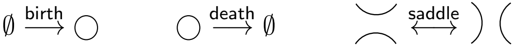
- Categorical or algebraic descriptions for singular knots are obtained by imposing a height function on the plane of the projection.
- $\Longrightarrow$ three moves that take into account the height function; these moves do not affect the topology of the diagram.





Singular knot cobordisms

- Two singular knots are cobordant if they are related via
  - singular knot isotopy, together with
  - Morse modifications:



- The surface connecting two cobordant singular knots is called a singular knot cobordism.

- We are interested in studying singular knot cobordisms in 4-space up to isotopy (rel. boundary).

Diagrams for singular knot cobordisms

- A singular knot cobordism can be studied diagrammatically via a projection in 3-space.
- A diagram of a singular knot cobordism is a projection of that from $\mathbb{R}^4$ into $\mathbb{R}^3$. **We fix a height function on $\mathbb{R}^3$.**
- Such a diagram can be cut by planes that are perpendicular to the fixed direction in $\mathbb{R}^3$.
- $\Longrightarrow$ movie description for a singular knot cobordism.

Movies for singular link cobordisms

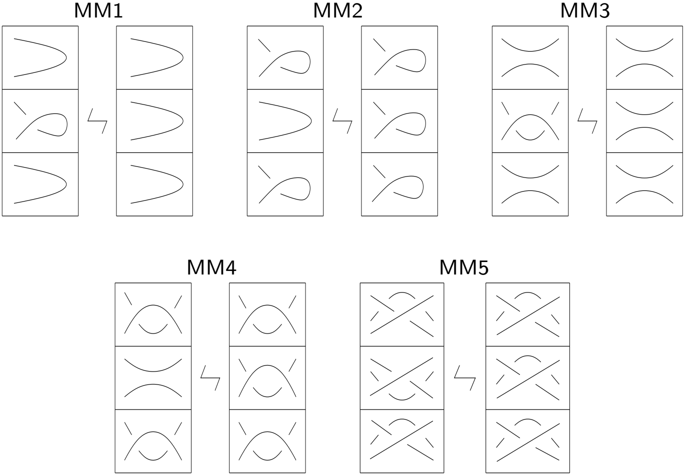
- A movie of a singular knot cobordism is a sequence of singular link diagrams, called stills.
- Two consecutive stills in a movie of a singular knot cobordism are diagrams that differ by either an extended Reidemeister move or a Morse modification.

Movie moves

Question: What are the moves for movies that describe isotopic singular knot cobordisms?

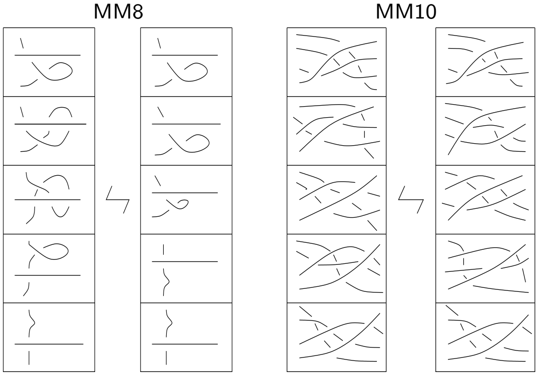
- Just as singular knots contain classical knots as a subset, singular knot cobordisms include classical knot cobordisms in 4-space.
- Therefore, our needed moves include all of the 15 Carter-Saito movie moves for classical knot cobordisms
- The Carter-Saito moves are extensions of the 7 moves by Roseman for diagrams of knot cobordisms - without a height function on $\mathbb{R}^3$.

Movie parametrizations of the Roseman moves:



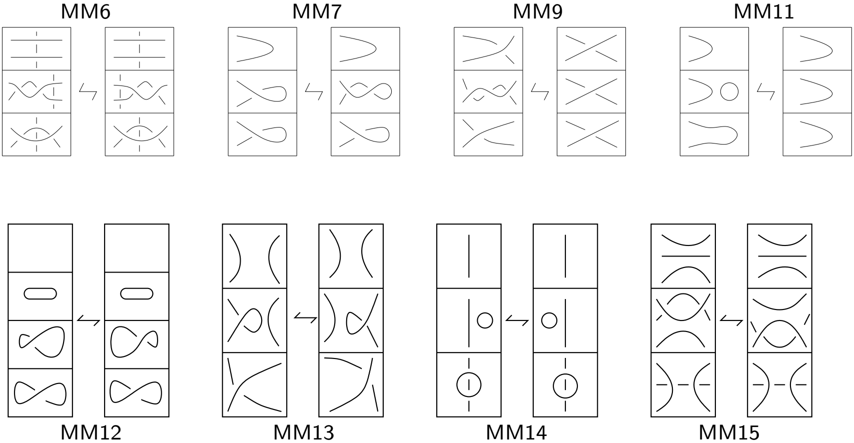
The Carter-Saito movie moves for knotted surfaces

Movie parametrizations of the Roseman moves (continued):

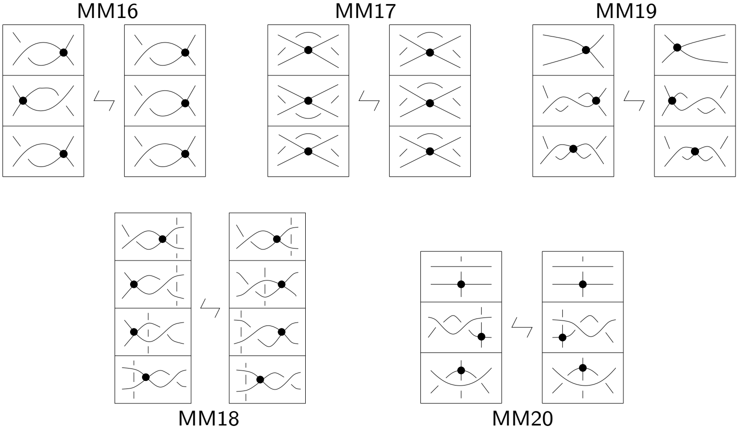


The Carter-Saito movie moves for knotted surfaces

The Carter-Saito augmented set of movie moves (when there is a height function in the 3-space of the projection); these moves do not affect the topology of the projection in $\mathbb{R}^3$:



Additional movie moves for singular knot cobordisms



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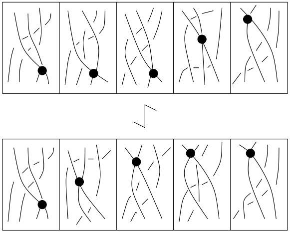
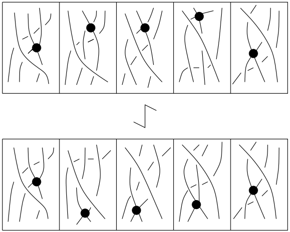
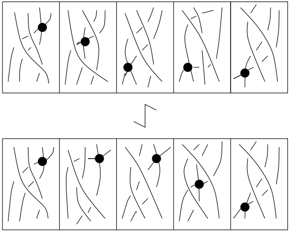


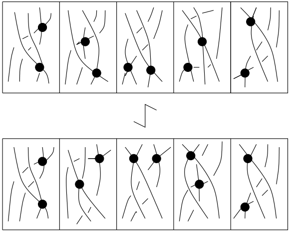
MM21



MM22



MM23



MM24

Movie-move theorem

Theorem (C.)

Two movie presentations represent isotopic singular knot cobordisms if and only if they are related to each other by a finite sequence of the moves for knotted surfaces (the 15 Carter-Saito movie moves) <u>*plus*</u> *the 9 movie moves in the previous two slides or interchanges of the levels of distant critical points.*

Remark. It is not hard to see that the $15 + 9$ movie moves represent isotopic singular knot cobordisms.

Movie-move theorem

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Remark. It is not hard to see that the $15 + 9$ movie moves represent isotopic singular knot cobordisms. The hard part of the proof is to show that the proposed list of movie moves is sufficient to relate any diagrams of isotopic singular knot cobordisms.

Theorem (C.)

Two movie presentations represent isotopic singular knot cobordisms if and only if they are related to each other by a finite sequence of the moves for knotted surfaces (the 15 Carter-Saito movie moves) plus the 9 movie moves in the previous two slides or interchanges of the levels of distant critical points.

Remark. It is not hard to see that the $15 + 9$ movie moves represent isotopic singular knot cobordisms. The hard part of the proof is to show that the proposed list of movie moves is sufficient to relate any diagrams of isotopic singular knot cobordisms. For that, we need to study the projection of an isotopy on a plane.

The philosophy behind the movies moves

Movie moves correspond to:

- (A.) critical points of the multiple point strata–where the isotopy direction provides a height function;
- (B.) transverse intersections in 4-space (between two strata);
- (C.) changes involving fold lines and cusps; these moves do not affect the topology of the diagram, and are needed only when we impose a height function in the 3-space of the projection.

That is, we look at local type of changes in one higher dimension as an isotopy of singular knot cobordisms.
One needs to make sure that the listed movie moves exhaust all of the possible critical behaviors in the (A.) - (C.) categories above.

Singular strata for diagrams of singular knot cobordisms

The 0-dim'l singular points on a given diagram of a singular knot cobordisms are:

- the branch points and crossing-vertex points that result from an $R1$ and $R5$ moves, respectively;
- triple points and singular intersection points that result from an $R3$ and $R4$ moves, respectively;

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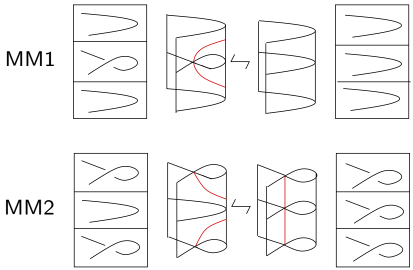
The 1-dim'l singular sets of the diagram of a cobordism are:

- the double-point arcs produced by transverse self-intersections;
- the edges of the cobordism.

Remarks

- During an isotopy, the 0-dim'l singular points and 1-dim'l singular sets trace 1-dim'l and 2-dim'l sets, respectively, and we need to examine their critical points.
- Since the topology of a cobordism is unchanged during an isotopy, the movie moves do not correspond to critical points for the edge set of a cobordism.

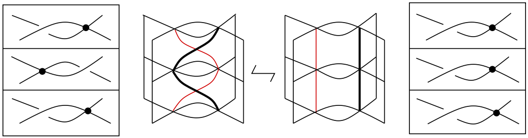
(A1.) Critical points of the branch-point set



- The movie moves *MM*1 and *MM*2 correspond to a critical point for the branch-point set.
- There is an optimum or a saddle on the set of self-intersection.

(A1.) Critical points of the crossing-vertex point set

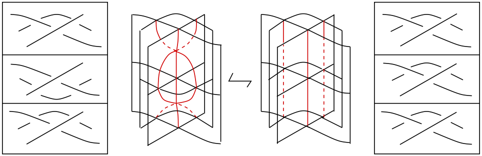
MM16



- The movie move *MM*16 corresponds to a critical point (optimum) of the crossing-vertex set.
- In the isotopy direction, a non-degenerate critical point corresponds to the annihilation/creation of a pair of crossing-vertex points.

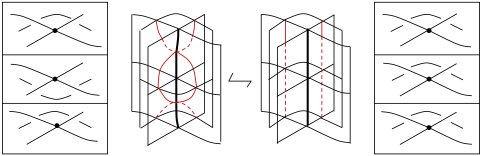
(A1.) Critical points of the triple-point set

MM5



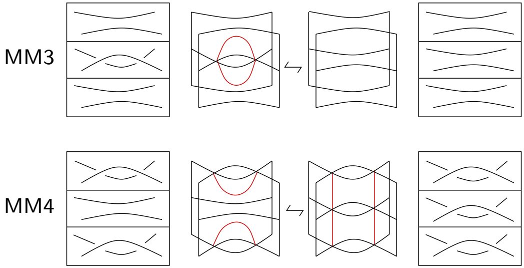
- The movie move $MM5$ corresponds to a critical point (optimum) of the triple-point set.
- In the isotopy direction, a non-degenerate critical point corresponds to the annihilation/creation of a pair of triple points.

(A1.) Critical points of the singular intersection-point set

MM17 

- The movie move *MM*17 corresponds to a critical point (optimum) of the singular intersection-point set.
- In the isotopy direction, a non-degenerate critical point corresponds to the annihilation/creation of a pair of singular intersection points.

(A2.) Critical points of the double-point set



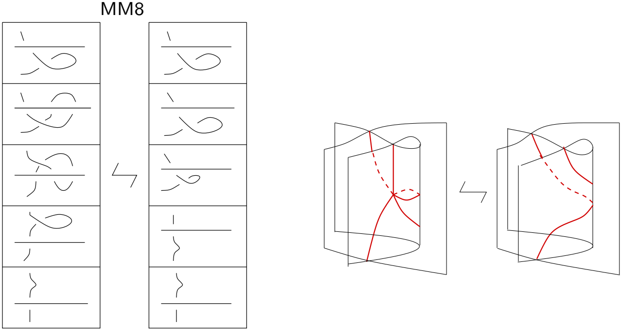
- The double-point set evolves to a surface during an isotopy.
- The critical points of this surface, namely optima and saddles, correspond to the movie moves *MM*3 and *MM*4, respectively.

(B.) Transverse intersections

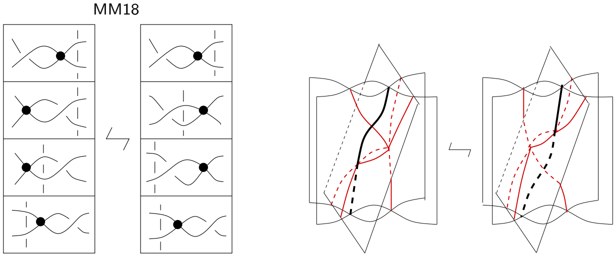
- The next set of movie moves deal with transverse intersections in the 4-dimensional space-time (the 3-space of the projection times the isotopy direction).
- A transverse intersection in 4-space occurs between
  - a 1-dim'l arc and a 3-dim'l solid (type (3-1)), or
  - between two 2-dim'l sets (type (2-2)).

The type (3-1) intersections are as follows:

- (3-1.B) A branch point passing through a transverse sheet:



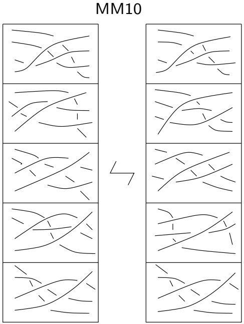
- (3-1.CV) A crossing-vertex point passing through a transverse sheet:

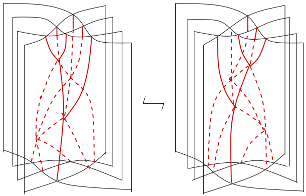


This move corresponds to an *isolated point* created as the transverse intersection between a crossing-vertex point set and a 3-dim'l set.

(B.) Transverse intersections – type (3-1)

- (3-1.T) A triple point passing through a transverse sheet:



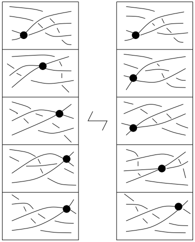


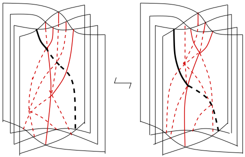
The triple point evolves in space-time to become a 1-dim'l arc and the transverse sheet evolves into a 3-dim'l solid; their transverse intersection in 4-space is an *isolated quadruple point*.

(B.) Transverse intersections – type (3-1)

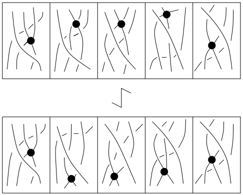
- (3-1.SI) A singular intersection point passing through a transverse sheet:

MM21

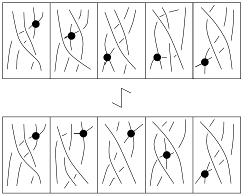




MM22



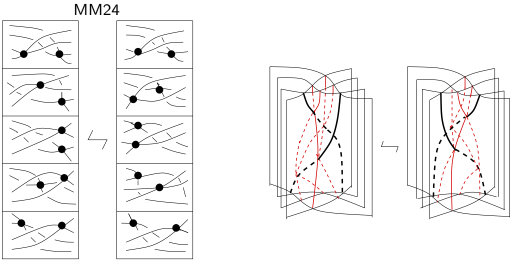
MM23



B. Transverse intersections – type (2-2)

The type (2-2) intersections are as follows:

- (2-2.XX) The intersection between two edges of the cobordism:

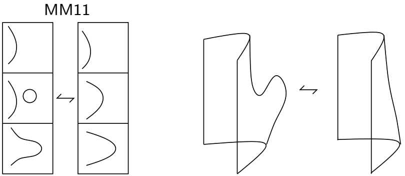


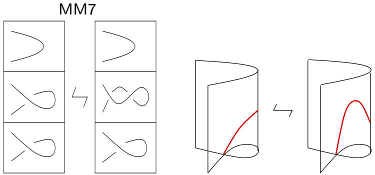
- (2-2.DD) The intersection between two double-point arcs; this is another way to interpret the movie move $MM10$.
- (2-2.XD) The intersection between an edge and a double-point arc; this is another way to interpret the movie moves $MM21$ through $MM23$.

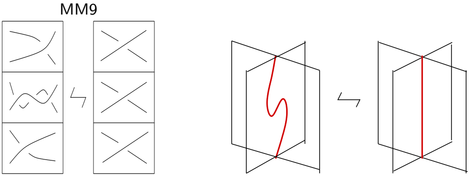
(C.) Movie moves involving fold lines and cusps

The remaining isotopy moves involve fold lines and cusps, and *do not affect the topology of the projection* in 3-space of a cobordism.

- (C1.) Changes that involve cusps:





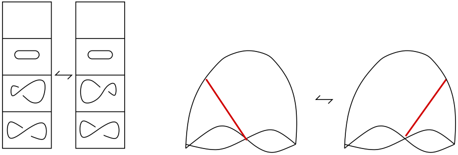


(C.) Movie moves involving fold lines and cusps

(C2.) Changes that affect the relative position of double-point arcs in relation to an optimum or saddle in a facet cobordism:

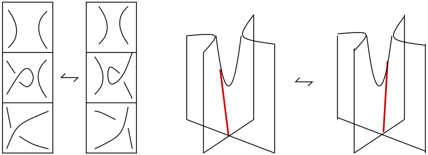
- a branch point may pass through an optimum on a fold line:

MM12



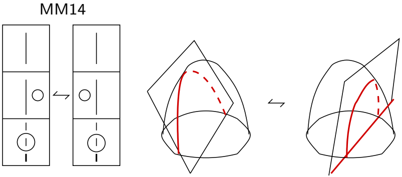
- or a branch point may pass through a saddle on a fold line:

MM13

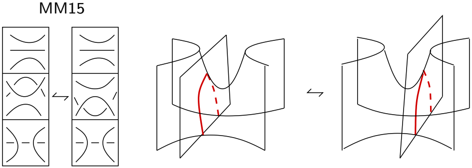


(C2.) Similarly,

- a double-point arc may pass over an optimum on a fold line:



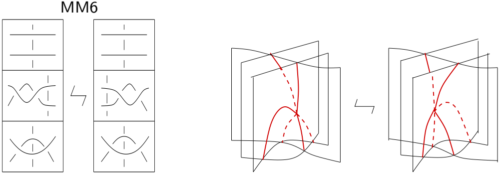
- or a double-point arc may pass through a saddle on a fold line:



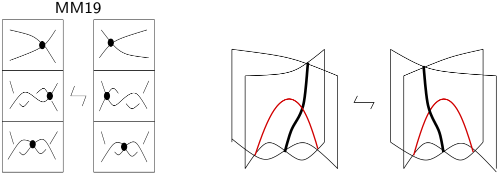
(C.) Movie moves involving fold lines and cusps

(C3.) Changes that affect the relative position of singular points in relation to the optimum point of a double-point arc:

- A triple point can be pushed over an optimum point of the double-point arc:



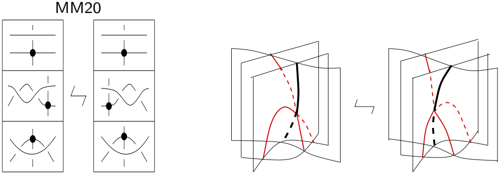
- A crossing-vertex point can pass through an optimum on a double-point arc:



(C.) Movie moves involving fold lines and cusps

(C3.) Similarly,

- A singular intersection point can be moved pass an optimum on a double-point arc:



THANK YOU!