

Book Representations of Complete Graphs

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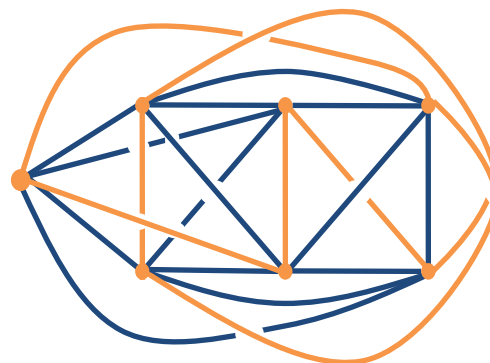
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Outline

- What are book representations of graphs?
 - Bounds on sheet number, edges per sheet
 - Book representations of K_5
- Knotted cycles in book representations
 - Arc presentation of knot
- When are two book representations equivalent?
- All book representations with 4 or less sheets
- Maximal sheet book representations
- Related open questions

Knotted cycles in complete graphs

- Every spatial embedding of K_7 contains at least one nontrivial knotted Hamiltonian cycle.
(Conway&Gordon, 1983)



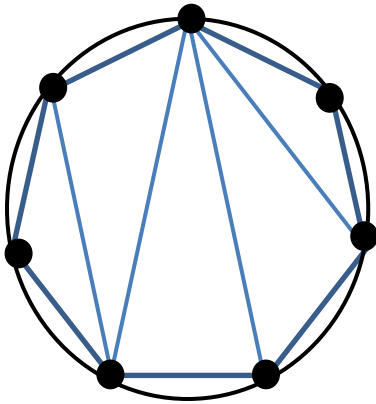
Open question 1: What is the smallest possible number of knotted cycles in an embedding of K_n ?

Open question 2: What is the smallest possible maximum crossing number for all cycles in an embedding of K_n ?

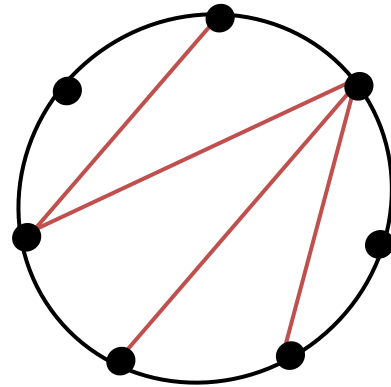
Book Representations

In book representations:

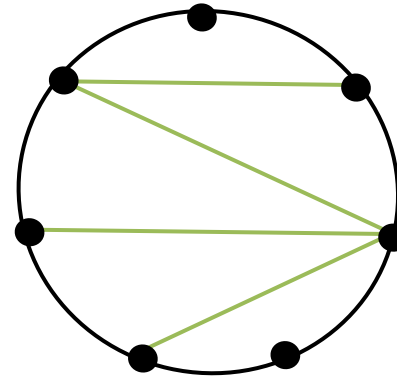
- Vertices lie on a circle
- Edges are chords on internally disjoint topological disks
- No edge crosses itself
- No adjacent edges cross
- Edges occur in layers



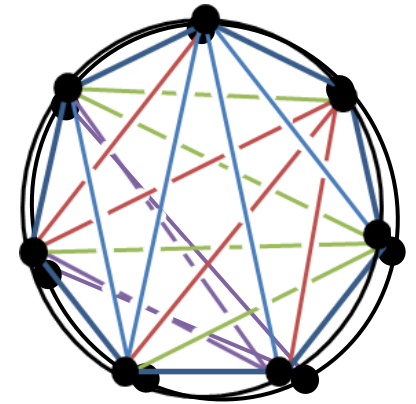
Sheet 1



Sheet 2



Sheet 3



Sheet 4

Edges in a Book Representation

Label the vertices $1, 2, 3, \dots, n$ traveling clockwise around the circle:

Let $a < b, a < c$ and $c < d$.

Edges ab and cd intersect if and only if $c < b < d$.

For $a < b$, we say ab is a j -edge if

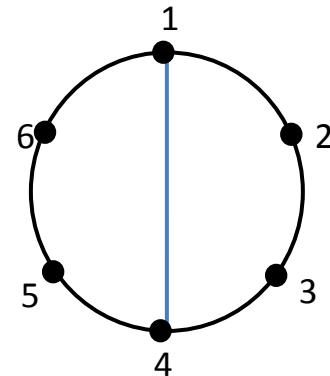
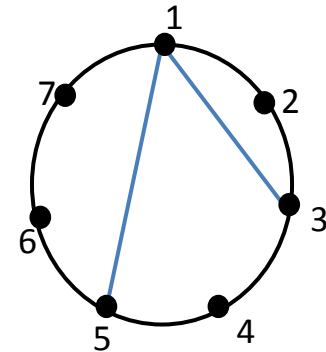
$$j = \text{length}(ab) \equiv \min((b - a), (a - b) \bmod n).$$

The number of edges of maximal length depends on whether n is even or odd:

- In K_{2m} the total number of m -edges is m .
- In K_{2m+1} the total number of m -edges is $2m + 1$.
- For $j < m$, the total number of j -edges is n .

To get a book representation:

- Assign a sheet to each edge with length at least 2.
- Edges in the same sheet cannot intersect.



Minimum/Maximum Sheet Number

- The **sheet number** of an embedding is the smallest possible number of sheets among all ambient isotopic book representations.
- Minimum sheet number:
Every m -edge must be on a separate sheet for K_{2m} and at most two m -edges are on each sheet for K_{2m+1} , giving:

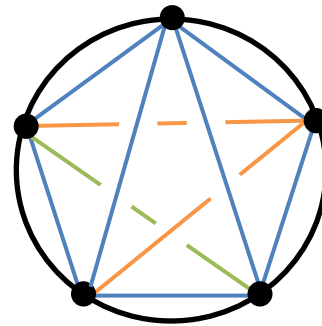
$$\left\lceil \frac{n}{2} \right\rceil$$

- The Canonical Book Representation has minimal sheet number (Otsuki).
- Maximum sheet number:
Each edge of length 2 or more must be on a separate sheet, giving:
$$\frac{n(n-1)}{2} - n = \frac{n(n-3)}{2}$$
- We constructed a family of maximal sheet book representations.

Book Representations for K_5

- **Minimum: 3 sheets**

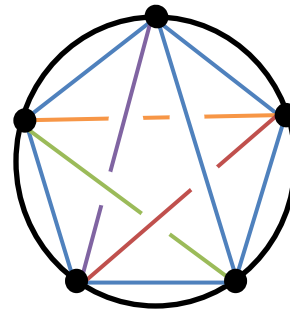
- Up to ambient isotopy, only one embedding.
- No knotted cycles.



Sheet	Edges
1	13, 14
2	24, 25
3	35

- **Maximum: 5 sheets**

- Up to ambient isotopy, two embeddings—this one, and its mirror.
- Cycle 13524 is a trefoil.



Sheet	Edges
1	13
2	24
3	35
4	14
5	25

- **There are no 4-sheet book representations.**

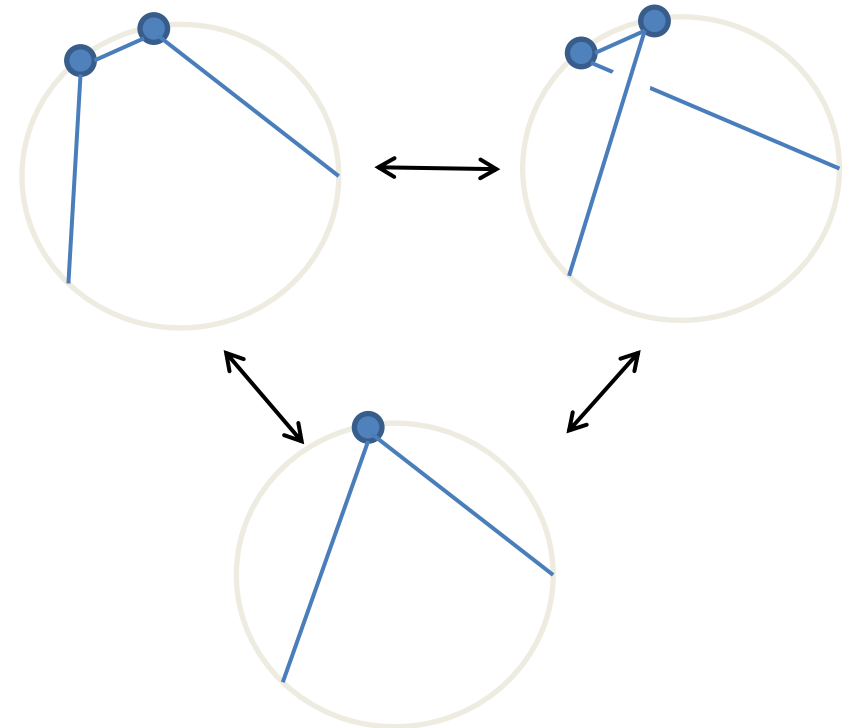
Cycles in Book Representations

- Canonical book representation:
 - Minimizes knotted 7-cycles: exactly $\binom{n}{7}$ trefoils (Otsuki)
 - For $n \leq 11$, the number and type of knotted cycles is known (R., Politano)
- A cycle in any book representation gives an **arc presentation** of a knot; the **arc-index** $\alpha(K)$ is the minimum number of vertices in an arc presentation.
 - Every knot has an arc presentation (Cromwell).
 - For alternating knots, $\alpha(K) = c(K) + 2$ (Bae-Park; Morton-Beltrami).
 - For non-alternating knots, $\alpha(K) \leq c(K)$ (Jin-Park).
 - All prime knots with arc index up to 11 are known (Jin-Park).
- A knot is a cycle in some book representation of K_n if and only if $\alpha(K) \leq n$.
 - Trefoils are the only possible knotted cycles in book representations of K_5 .
 - Trefoils and figure-eight knots are the only possible knotted cycles in book representations of K_6 .

Arc presentation

Any two arc-presentations are related by a finite sequence of the following moves (Cromwell):

1. Vertex exchange: Adjacent vertices can be exchanged unless their edges are in interleaving sheets
2. Arc exchange: Non-intersecting edges in adjacent sheets can be exchanged.
3. Arc merge/divide: If endpoints of an edge are adjacent, the edge can be removed. Conversely, an edge can be inserted at any vertex.



Equivalence of Book Representations

The following moves give equivalent book representations:

- 1. Rotating vertices:** Increase or decrease all vertex labels by one, mod n .
- 2. Shifting sheets:** Increase or decrease all sheet numbers by one, mod s .
- 3. Vertex exchange:** If v and w are adjacent vertices, and if the sheet number of each edge incident to v is higher than the sheet number of any edge incident to w , then vertices v and w can be exchanged.
- 4. Edge move:** If an edge does not intersect any of the edges in an adjacent sheet, then it can be moved into that sheet.
- 5. Sheet insertion/deletion:** An empty sheet can be added or removed between any two sheets.

Note that the mirror image is formed by reversing the order of the sheets.

Example:

By #3:

By #2:

By #4:

By #1:

Sheet	Edges
1	13, 14, (15)
2	(23), 24, 25
3	35

Sheet	Edges
1	(23), 24, 25
2	13, 14, (15)
3	35

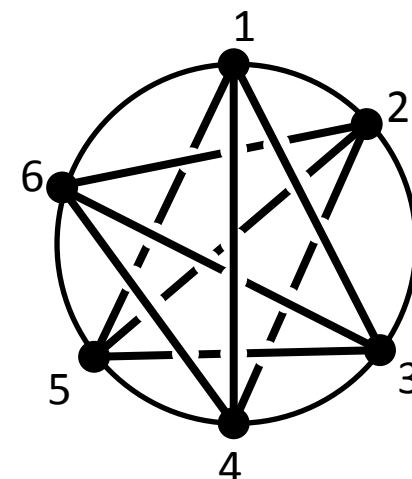
Sheet	Edges
1	35
2	24, 25
3	13, 14

Sheet	Edges
1	35, 25
2	24
3	13, 14

Sheet	Edges
1	14
2	35, 13
3	24, 25

Minimal book representation of K_6

- We have:
 - 3 sheets total
 - 6 edges of length 2 (two per sheet)
 - 3 edges of length 3 (1 per sheet)
- We can assume edges 13 and 14 are in Sheet 1 (shift, rotate if necessary).
- Only two valid ways to fill the sheets:



Sheet	Edges
1	13, 14, 46
2	26, 35, 36
3	15, 24, 25

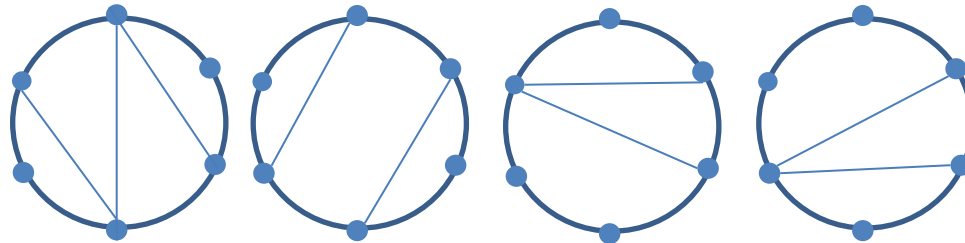
Sheet	Edges
1	13, 14, 46
2	15, 24, 25
3	26, 35, 36

These are mirror images. Neither embedding has knotted cycles. In fact, they are ambient isotopic—this gives only one minimal representation.

K_6 on 4 sheets

- 9 edges to distribute—either 3, 2, or 1 per sheet.
- We can assume that edges 13, 14 are in sheet 1.
- Since 15 and 46 intersect, one of them must be in a sheet adjacent to sheet 1, so we can assume sheet 1 consists of 13, 14, 15 or 13, 14, 46.
- This yields 127 valid sheet assignments, but all are equivalent to either the one below, its mirror image, or the minimal book representation:

Sheet	Edges
1	13, 14, 46
2	15, 24
3	26, 36
4	25, 35



Cycle 136425 is a trefoil. All others are unknots.

Minimal book representations of K_7

- Every book representation of K_7 with 4 sheets is equivalent to, or a mirror image of, one of the following:

Sheet	Edges
1	13, 14, 15, 57
2	16, 24, 25, 26
3	27, 35, 36, 37
4	46, 47

Canonical book rep.
One knotted cycle:
1357246 is a (left)
trefoil.

Sheet	Edges
1	13, 14, 15, 57
2	27, 36, 37, 46
3	16, 25, 26, 35
4	24, 47

Five trefoils—4
left, 1 right.

Sheet	Edges
1	13, 14, 15, 57
2	36, 37, 46
3	16, 25, 26, 35
4	24, 27, 47

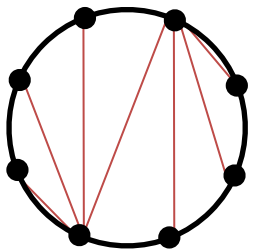
Three trefoils.

Sheet	Edges
1	14, 15, 24, 57
2	27, 36, 37, 46
3	16, 25, 26, 35
4	13, 47

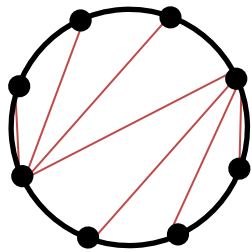
Seven trefoils.

Minimal book representations of K_8

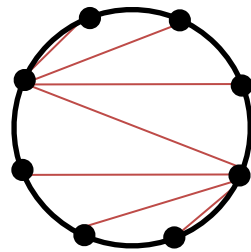
(Left) canonical book representation



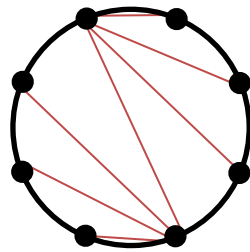
Sheet 1



Sheet 2



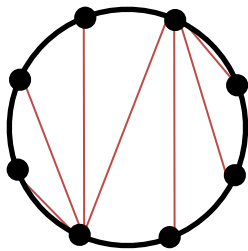
Sheet 3



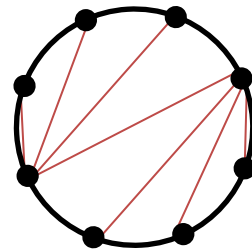
Sheet 4

- 21 knotted Hamiltonian cycles
- 8 knotted 7-cycles
- All knotted cycles are left-handed trefoils

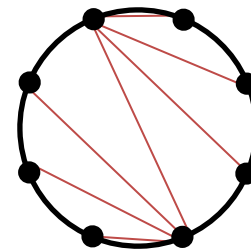
Exchange last two sheets:



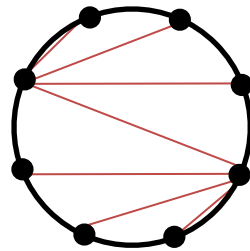
Sheet 1



Sheet 2



Sheet 3



Sheet 4

- 23 knotted Hamiltonian cycles
 - 11 left-handed trefoils
 - 11 right-handed trefoils
 - 1 figure-eight knot
- 8 knotted 7-cycles

Maximal book representations of K_6

- Each edge is in its own sheet, so must intersect the edges in the above and below sheets.
- All maximal book representations are equivalent by rotation to one of these or their mirror images:

Sheet	Edge
1	14
2	25
3	36
4	24
5	13
6	26
7	15
8	46
9	35

- The length 3 edges are consecutive.
- Five Hamiltonian cycles are knotted—three right trefoils and two left trefoils.

Sheet	Edge
1	14
2	25
3	36
4	24
5	13
6	26
7	15
8	46
9	35

- The length of the edges are in a 3-2-2-3-2-2-3-2-2 pattern.
- Nine Hamiltonian cycles are knotted—six right trefoils and three figure-eight knots.

Algorithms for maximal book representations

Start with the length m (maximal length edges), then length $m-1$, and so on to length 2.

EVEN CASE: Use ALL the length m edges first:

ODD CASE: Use one edge of each length:

$$K_{2m}$$

For $1 \leq i \leq m$,

Join i to $i + m$.

For $1 \leq j \leq m - 2$,

for $1 \leq i \leq 2m$,

Join $m - i$ to

$(m - i) + (m - j)$.

For K_8 , this gives 419 knotted Hamiltonian cycles, including knot types

$3_1, 4_1, 5_1, 5_2, 6_1, 8_{19}, 10_{124}$.

$$m = 4$$

Sheet	Edge
1	15
2	26
3	37
4	48
5	36
6	25
...	
19	57
20	46

$$K_{2m+1}$$

For $0 \leq i \leq 2m$,

for $0 \leq j \leq m - 2$,

Join $1 + 4i - j$ to

$1 + 4i - j + (m - j)$.

For K_7 , this gives 57 knotted Hamiltonian cycles, 43 trefoils and 14 figure-eight knots.

$$m = 4$$

Sheet	Edge
1	15
2	39
3	18
4	59
5	47
6	35
...	
26	58
27	46

Open questions

- Which knots can be realized as knotted cycles in s -sheet book representations of K_n ?
- For a given knot, what is the smallest s so that the knot appears as a cycle in an s -sheet book representation for some K_n ?
- How many distinct book representations are there for K_n ?
- What is the smallest n so that a book representation of K_n must contain a knot with crossing number at least c ?
 - When $c = 3$, we know $n = 7$.
 - Conjecture: $n = 9$ for $c = 4$.
- What is the minimum number of knotted cycles possible in an embedding of K_n when $n \geq 8$?
 - The canonical book representation gives an upper bound.
 - An embedding of K_8 has at least 15 knotted cycles (Abrams-Mellor).

Thank you!

References

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