Coloring Spatial Graphs
(Preliminary Report)

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This project is joint work with several of my students:

- Terry Kong
- Alec Lewald
- Vadim Pigrish

Terry and Alec are continuing the project with me this summer.

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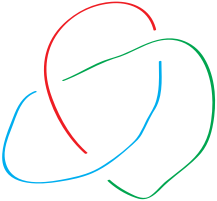
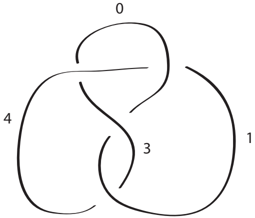
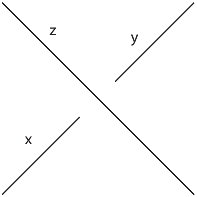


tricoloring



mod 5 coloring



$2z - x - y = 0 \pmod{p}$

Knot determinant

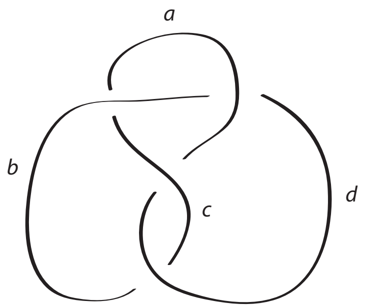


figure-8 knot

$$\begin{array}{ccccc} 2a & -b & & -d & =0 \\ -a & +2b & -c & & =0 \\ -a & & +2c & -d & =0 \\ & -b & -c & +2d & =0 \end{array}$$

$$\begin{bmatrix} 2 & -1 & 0 & -1 \\ -1 & 2 & -1 & 0 \\ -1 & 0 & 2 & -1 \\ 0 & -1 & -1 & 2 \end{bmatrix}$$

$$\det(K) = \begin{vmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ -1 & 0 & 2 \end{vmatrix} = 5$$

Definition

The determinant of a diagram for a knot K with c crossings is the determinant of any $(c-1) \times (c-1)$ minor of the coefficient matrix for the system of crossing equations.

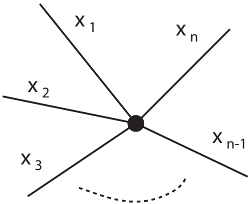
The determinant does *not* depend on which minor is used, or on the choice of diagram for K (i.e. it is invariant under Reidemeister moves).

Theorem

A knot K is p-colorable if and only if p divides $\det(K)$.

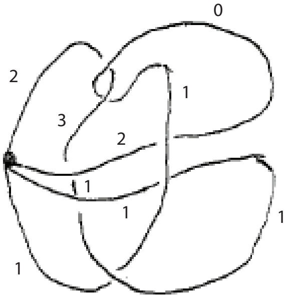
Colorings of graphs were first investigated by Ishii and Yasuhara (1997).

What relation is needed at the vertices?



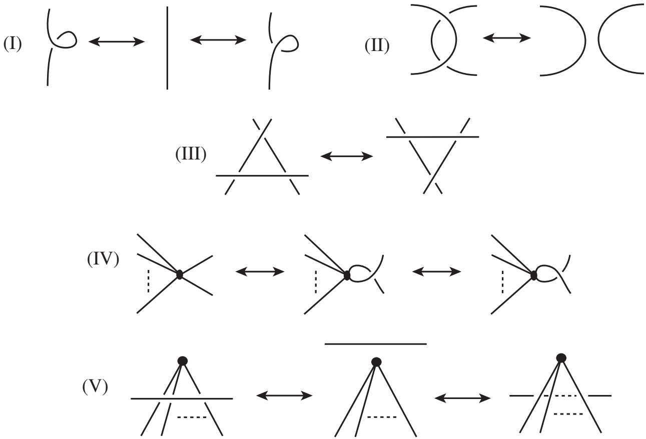
$x_1 - x_2 + x_3 - \ldots - x_n = 0$

Here is a 4-coloring of a graph.



(Graphs must have all vertices of even degree.)

Reidemeister moves for graphs



$$\begin{bmatrix} -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 2 & -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 & -1 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 2 & -1 \\ 1 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \\ x_8 \\ x_9 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Matrix has $c + e$ columns and $c + v$ rows, where c, e, v are the numbers of crossings, edges, vertices. p-colorings are non-trivial solutions (mod p) where at least two colors used.

Observation: Any graph with $e > v$ is colorable with at least $e - v$ linearly independent non-trivial colorings.

Definition

The determinant of an embedded graph is the greatest common divisor of the determinants of all $(c + v - 1) \times (c + v - 1)$ minors.

Theorem

The determinant of a graph is invariant under the graph Reidemeister moves.

Theorem

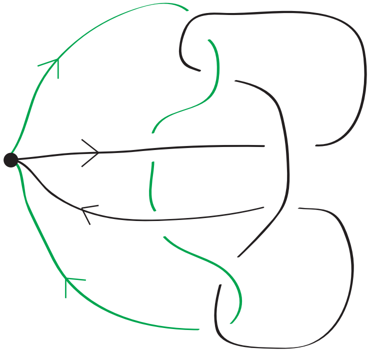
An embedded graph (with all vertices having even degree) has $e - v + 1$ independent non-trivial colorings mod p if and only if p divides the determinant.

Some properties of the determinant of a graph:

- $\det(G \vee G') = \det(G)\det(G')$
- If G' is the result of contracting an edge of G, then $\det(G') = \det(G)$
- (Conjecture) $\det(G \# G') = \det(G)\det(G')$

Colorings were considered from a very general (algebraic topological) perspective by McAtee, Silver and Williams (2001). Some of our results are implied by their paper, but our approach is much more elementary.

Example



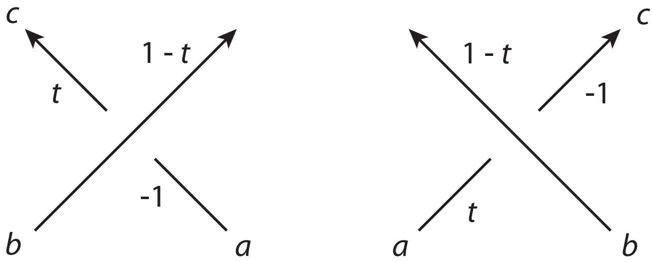
$\det(G) = 5$

This implies G has at least two independent 5-colorings.

Note that both loops of the bouquet are unknotted.

Alexander polynomial for a knot

Like the knot determinant, the Alexander polynomial for a knot begins by defining a linear relation at each crossing of the knot diagram. In this case, the diagram is *oriented*.



$$(1 - t)b - a + tc = 0$$

$$(1 - t)b + ta - c = 0$$

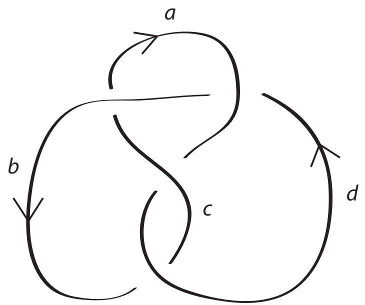


figure-8 knot

$$\begin{array}{llllll}
(1-t)a & -b & & +td & = 0 \\
-a & +(1-t)b & +tc & & = 0 \\
-a & & +(1-t)c & +td & = 0 \\
 & -b & +tc & +(1-t)d & = 0
\end{array}$$

$$\begin{bmatrix} 1-t & -1 & 0 & t \\ -1 & 1-t & t & 0 \\ -1 & 0 & 1-t & t \\ 0 & -1 & t & 1-t \end{bmatrix}$$

$$\Delta_K(t) = \begin{vmatrix} 1-t & -1 & 0 \\ -1 & 1-t & t \\ -1 & 0 & 1-t \end{vmatrix} = -t^3 + 3t^2 - t = -t(t^2 - 3t + 1)$$

$\Delta_K(t)$ is well-defined up to multiplication by $\pm t^k$, so we usually say the polynomial for the figure-8 knot is $t^2 - 3t + 1$.

Observe: $\Delta_K(-1) = \det(K)$.

Alexander polynomial for a graph?

It is natural to try to extend the Alexander polynomial to graphs.

This was first done by Kinoshita in 1958, using Fox's free calculus, but has received surprisingly little attention since. It was also done by McAtee, Silver and Williams (2001) using algebraic topology.

Our goal is to provide an elementary, combinatorial, formulation, which we hope to use to prove useful properties of the polynomial, and to compute it for a range of examples.

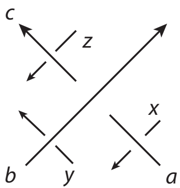
The first question is how to define the appropriate relation at the vertices. For this, we review the fact that the Alexander polynomial of a knot arises from a representation of the fundamental group.

If x_i is a generator of the Wirtinger presentation of the fundamental group (corresponding to an arc a_i of the diagram), then the representation sends:

$$x_i \longmapsto \begin{pmatrix} \sqrt{t} & a_i \\ 0 & \frac{1}{\sqrt{t}} \end{pmatrix}$$

$$x_i^{-1} \longmapsto \begin{pmatrix} \frac{1}{\sqrt{t}} & -a_i \\ 0 & \sqrt{t} \end{pmatrix}$$

Then the Alexander relations arise from the Wirtinger relations at each crossing.

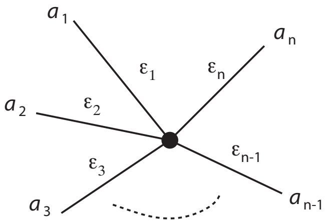


$$xyz^{-1}y^{-1} \longmapsto \begin{pmatrix} \sqrt{t} & a \\ 0 & \frac{1}{\sqrt{t}} \end{pmatrix} \begin{pmatrix} \sqrt{t} & b \\ 0 & \frac{1}{\sqrt{t}} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{t}} & -c \\ 0 & \sqrt{t} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{t}} & -b \\ 0 & \sqrt{t} \end{pmatrix}$$

$$= \begin{pmatrix} 1 & -\sqrt{t}(-a + (1-t)b + tc) \\ 0 & 1 \end{pmatrix}$$

So $xyz^{-1}y^{-1} = 1$ implies $-a + (1-t)b + tc = 0$.

For a graph, there is an additional Wirtinger relation at each vertex. Each arc a_i at the vertex has a sign ε_i, where $\varepsilon_i = 1$ if the edge is directed outwards, and $\varepsilon_i = -1$ if the edge is directed inwards.



If x_i is the Wirtinger generator for arc a_i, then:

$$x_1^{\varepsilon_1} x_2^{\varepsilon_2} x_3^{\varepsilon_3} \ldots x_{n-1}^{\varepsilon_{n-1}} x_n^{\varepsilon_n} \longmapsto \begin{pmatrix} \sqrt{t^{\varepsilon_1}} & \varepsilon_1 a_1 \\ 0 & \sqrt{t^{-\varepsilon_1}} \end{pmatrix} \cdots \begin{pmatrix} \sqrt{t^{\varepsilon_n}} & \varepsilon_n a_n \\ 0 & \sqrt{t^{-\varepsilon_n}} \end{pmatrix}$$

$$x_1^{\varepsilon_1} x_2^{\varepsilon_2} x_3^{\varepsilon_3} \dots x_{n-1}^{\varepsilon_{n-1}} x_n^{\varepsilon_n} \longmapsto \begin{pmatrix} \sqrt{t^{\varepsilon_1}} & \varepsilon_1 a_1 \\ 0 & \sqrt{t^{-\varepsilon_1}} \end{pmatrix} \cdots \begin{pmatrix} \sqrt{t^{\varepsilon_n}} & \varepsilon_n a_n \\ 0 & \sqrt{t^{-\varepsilon_n}} \end{pmatrix}$$

$$= \begin{pmatrix} t^{\frac{1}{2}\sum_{i=1}^{n} \varepsilon_i} & \sum_{i=1}^{n} \varepsilon_i a_i t^{\frac{1}{2}\left(\sum_{j=1}^{i-1} \varepsilon_j - \sum_{j=i+1}^{n} \varepsilon_j\right)} \\ 0 & t^{-\frac{1}{2}\sum_{i=1}^{n} \varepsilon_i} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

We require $\sum_{i=1}^{n} \varepsilon_i = 0$. Then
$\frac{1}{2}\left(\sum_{j=1}^{i-1} \varepsilon_j - \sum_{j=i+1}^{n} \varepsilon_j\right) = \sum_{j=1}^{i-1} \varepsilon_j + \frac{1}{2}\varepsilon_i$, so:

$$\sum_{i=1}^{n} \varepsilon_i a_i t^{\left(\sum_{j=1}^{i-1} \varepsilon_j + \varepsilon_i/2\right)} = 0$$

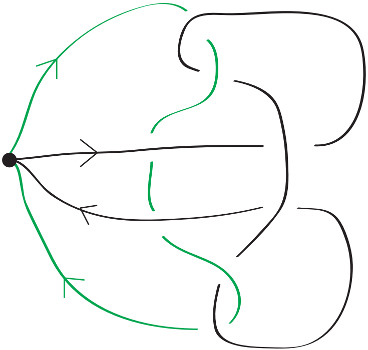
So this gives us a linear relation at each crossing and vertex of a graph, and we set up a matrix as we do for knots.

As with the determinant, the resulting matrix is *not* square.

Definition

The Alexander polynomial Δ_G of an embedded graph G is the (polynomial) greatest common divisor of the determinants of all $(c+v-1)\times(c+v-1)$ minors. (i.e. the generator of the polynomial ideal generated by all the minors)

As for knots, this is only well-defined up to multiplication by $\pm t^k$. With that caveat, it is invariant under the Reidemeister moves.



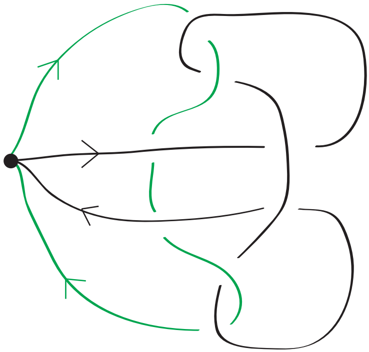
We will again consider this 2-bouquet graph, with the edges oriented as shown.

There are 9 rows in the Alexander matrix (8 crossings and 1 vertex), and 10 columns.

The matrix is on the next slide.

$$\begin{bmatrix} -1 & 0 & 0 & 0 & t & 0 & 1-t & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1-t & t & -1 & 0 & 0 & 0 \\ 0 & 1-t & 0 & 0 & t & 0 & 0 & -1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 1-t & t & 0 & 0 & 0 \\ 0 & 0 & 1-t & 0 & 0 & 0 & 0 & -1 & t & 0 \\ 0 & 0 & -1 & 0 & 0 & 1-t & 0 & 0 & 0 & t \\ 0 & 0 & 0 & 0 & 0 & t & 0 & 0 & 1-t & -1 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & t & 1-t \\ t^{1/2} & t^{-1/2} & -t^{-1/2} & -t^{1/2} & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The Alexander polynomial is $\Delta_G(t) = t^3 - 2t + 2$.



This graph has four orientations, each with an Alexander polynomial:

- $t^3 - 2t + 2$ (as shown)
- $t^2 - 3t + 1$ (reverse black)
- $t^2 - 3t + 1$ (reverse green)
- $2t^2 - 2t + 1$ (reverse both)

With graphs (unlike knots) the Alexander polynomial can sometimes distinguish a graph from its inverse.

Properties of the Alexander polynomial

- If G^* is the inverse of G (orientation of every edge reversed), then $\Delta_{G^*}(t) = \Delta_G(t^{-1})$.
- $\Delta_G(-1) | \det(G)$ (but may not be equal).
- $\Delta_{G_1 \vee G_2}(t) = \Delta_{G_1}(t)\Delta_{G_2}(t)$.
- If G' is the result of contracting an edge of G, then $\Delta_{G'}(t) = \Delta_G(t)$.
- (Conjecture) $\Delta_{G_1 \# G_2}(t) = \Delta_{G_1}(t)\Delta_{G_2}(t)$.

Further questions

- How do we extend to graphs with odd-valence vertices?
- What properties of the Alexander polynomial for knots carry over to graphs?
- Is there still a skein relation?
- How does the Alexander polynomial compare to the Yamada polynomial or other invariants of spatial graphs?
- A graph is *strongly* p-colorable if every edge at a vertex has the same color. How do we detect strong p-colorability?

References

- Y. Ishii and A. Yasuhara. Color Invariant for Spatial Graphs, *J. Knot Theory Ramif.*, v. 6 (1997), pp. 319-325
- S. Kinoshita. Alexander Polynomials as Isotopy Invariants, I, *Osaka Math. J.*, v. 10 (1958), pp. 263-271
- J. McAtee, D. Silver and S. Williams. Coloring Spatial Graphs, *J. Knot Theory Ramif.*, v. 10 (2001), pp. 109-120

Thank You

- Thank you all for coming to this talk.
- Any questions?
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