Graphs on 21 edges that are 2–apex

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Spatial Graphs Conference

Two Families

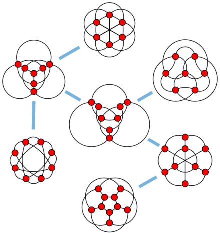
- Petersen Family
- Heawood Family
- I(K or C3L)
- MMN2A?

Proofs

- Case 1: $|V(G)| > 13$
- Proof of Proposition 2
- Case 2: $|V(G)| < 11$

Questions

Petersen Family



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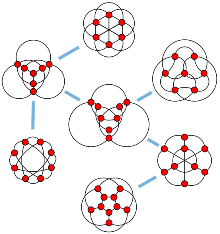

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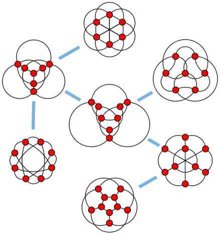

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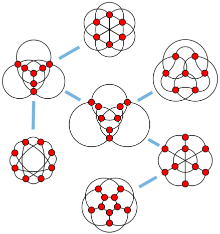

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Theorem[RST]: The Petersen family are precisely the minor minimal intrinsically linked graphs

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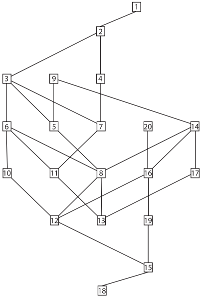
Theorem[LKLO,BM]: The KS graphs are the minor minimal intrinsically knotted graphs of 21 edges.

Heawood Family



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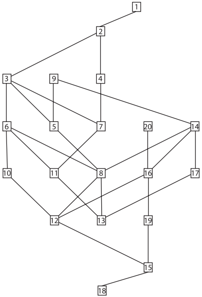

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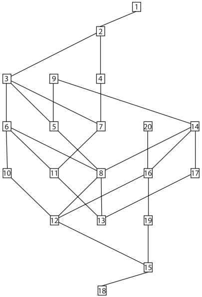

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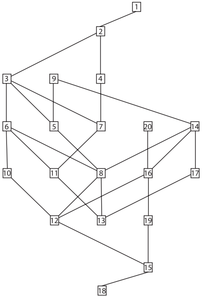

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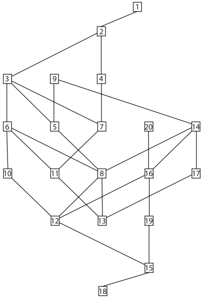

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How to characterise this family?

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Lemma[HNTY]: MMI(K or C3L) is preserved by $Y\Delta$

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Note that $K_{3,3} \sqcup K_{3,3} \sqcup K_{3,3}$ is MMN2A

Results

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Question: Does $Y\Delta$ preserve N2A on the set of graphs with 21 edges?

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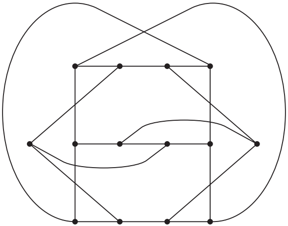

Figure: The Heawood graph, C_{14}

Suppose $|V(G)| = 14$. Want to argue that G is C_{14}.

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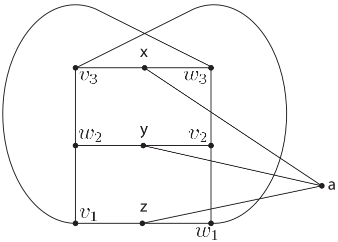

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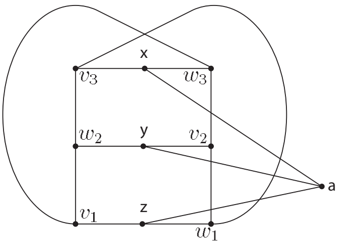

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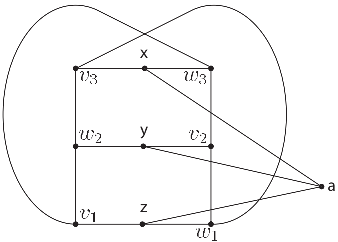

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$G - w_3$ is again of form shown above with b taking the place of a.

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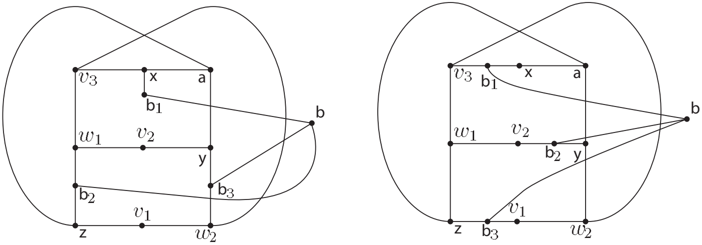

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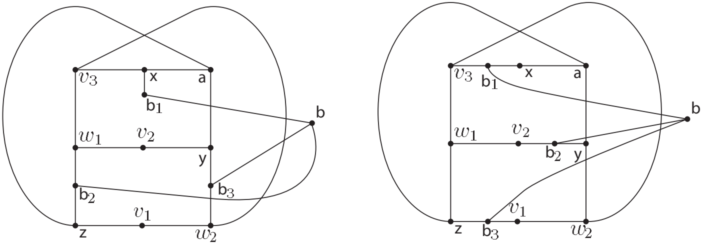

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In either case, adding in w_3 gives the Heawood graph C_{14}

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9 vertices: [M–] G is a Heawood (family) graph (+ isolated vertices) and $Y\Delta$ gives another Heawood graph

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H_2 = part of $G' - a, b$ outside triangle

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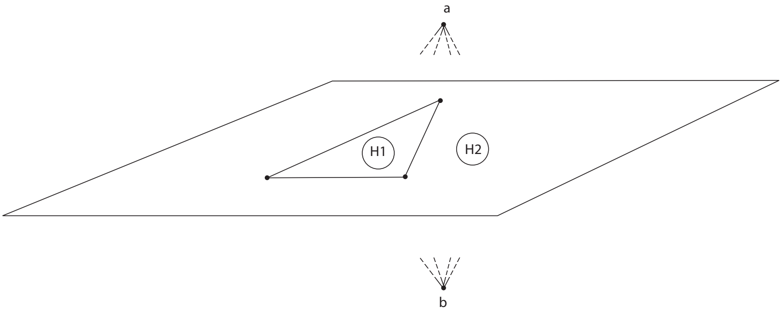

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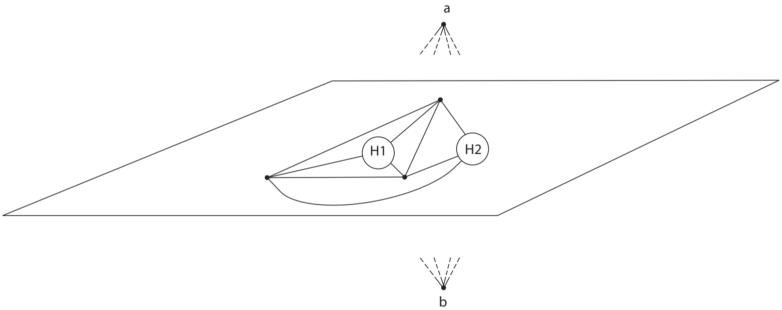

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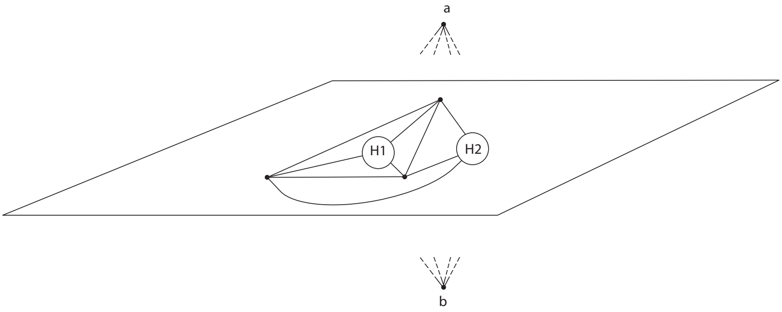

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Conclude that G' is N2A when $|V(G)| = 10$ to complete proof of Prop. 2

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Running through the cases, we conclude the proof of Proposition 1

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Question 3: What is the simplest G that is N2A but admits $Y\Delta$ move to G' that's not N2A?

An answer

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(Compare [LKLO,BM])

References

BM J.Barsotti & T.W.Mattman. Intrinsically knotted graphs with 21 edges. Preprint (arXiv).

HNTY R.Hanaki, R.Nikkuni, K.Taniyama, & A.Yamazaki. On intrinsically knotted or completely 3-linked graphs. *Pacific J. Math.* (2011).

KS T.Kohara and S.Suzuki. Some remarks on knots and links in spatial graphs. in *Knots 90, Osaka* (1992).

LKLO M.J.Lee, H.J.Kim, H.J.Lee, & S.Oh. Exactly fourteen intrinsically knotted graphs have 21 edges. Preprint (arXiv).

RST N.Roberston, P.Seymour & Thomas. Sachs's linkless embedding conjecture. *J. Combin, Theory Ser. B* (2004).