The conflict graph

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Definitions.

Given a subgraph (cycle), C, of a graph G, a *fragment* of C is a connected component of $G - C$, together with edges of attachment to C, or a chord of C.

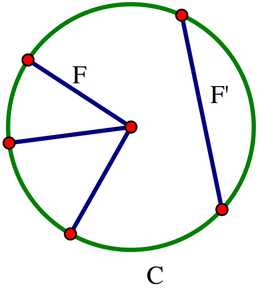


Figure: A cycle C and two different fragments.

Given a cycle C, two $C-$fragments A, B *conflict* if they have three common vertices of attachment to C or if there are four vertices v_1, v_2, v_3, v_4 in cyclic order on C such that v_1, v_3 are vertices of attachment of A and v_2, v_4 are vertices of attachment of B.

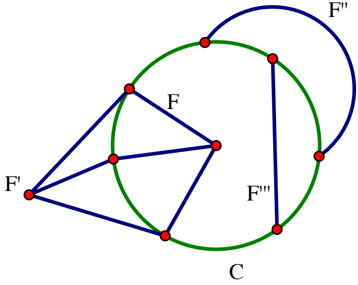
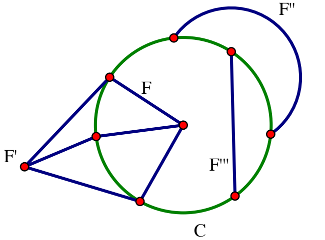


Figure: The fragments F and F' conflict, as do F'' and F'''.

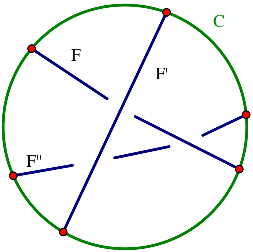
The *conflict graph* of C is a graph whose vertices are the $C-$fragments of G, with conflicting $C-$fragments adjacent.



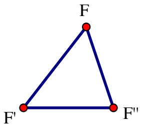
The conflict graph of C:

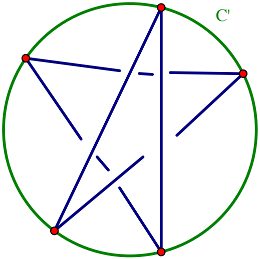
Tutte (1958) showed that G can be embedded in the plane if and only if the conflict graph of every cycle in G is bipartite. This can be proven using Kuratowski's theorem.

Note that the conflict graph of a Hamiltonian cycle in $K_{3,3}$ (K_5) is not bipartite:

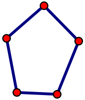


Conflict Graph of C:





Conflict Graph of C':



Q: Is there an analogous result for spatial graphs?

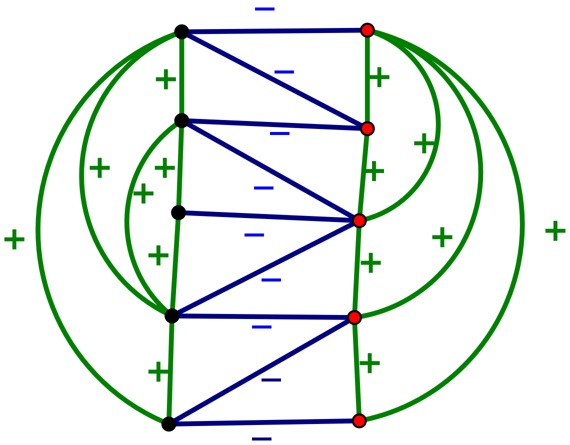
A *signed graph* $\Sigma = (G, \sigma)$ consists of a graph G and an edge signing $\sigma : E(G) \to \{+, -\}$.

The sign of a cycle $C = e_1 e_2 \cdots e_{n-1} e_n$ is obtained by multiplying the signs of its constituent edges:

$$\sigma(C) = \sigma(e_1)\sigma(e_2)\cdots\sigma(e_n)$$

A signed graph Σ is *balanced* if all of its cycles are positive. Otherwise, it is *unbalanced*.

In 1958, Harary showed that a signed graph is balanced if and only if its vertex set can be divided into two sets (one of which may be empty), X and Y, so that each edge between the sets is negative and each edge within the sets is positive.



In this way, balanced signed graphs are a generalization of bipartite graphs.

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But why should maximal planar subgraphs in a flat embedding be analogous to cycles in a planar embedding?

And what does it mean for fragments of a maximal planar subgraph to (anti-)conflict?

Established facts about maximal planar subgraphs:

1. A maximal planar subgraph of a 3-connected graph is connected. (Caveat: for the rest of the talk, we'll assume our maximal planar subgraphs are 2-connected.)
2. Given an S^2 embedding of a 2-connected graph, every region is bounded by a cycle.
3. The only possible fragments of a maximal planar subgraph are individual edges. That is, all of the vertices of the graph lie in the maximal planar subgraph.

Why are maximal planar subgraphs important?

Recall that a *flat* embedding of a graph G is a spatial embedding such that every cycle of G bounds a disk that has interior disjoint from the graph. We call such a disk a *panelling* disk.

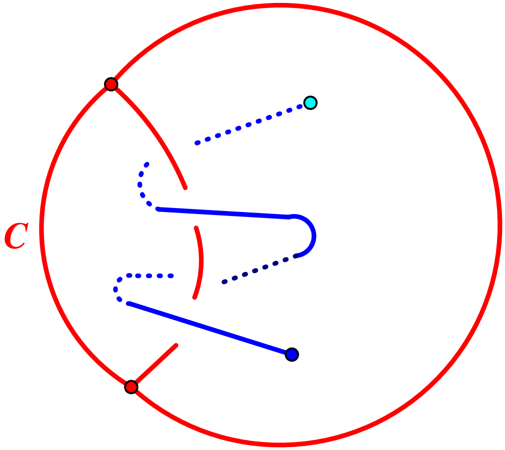
Wu (1992) showed that if G has a planar (S^2-)embedding, then for any given S^2-embedding of G and for every flat embedding of G, there exists a sphere in space that contains the S^2-embedding of G.

With a little more work, Wu's result further implies that if G has a flat embedding, and if P is a (2-connected) planar subgraph of G, then for a given flat embedding of G, there exists a sphere S that contains the induced embedding of P, and S is disjoint from the interiors of all edges not in P.

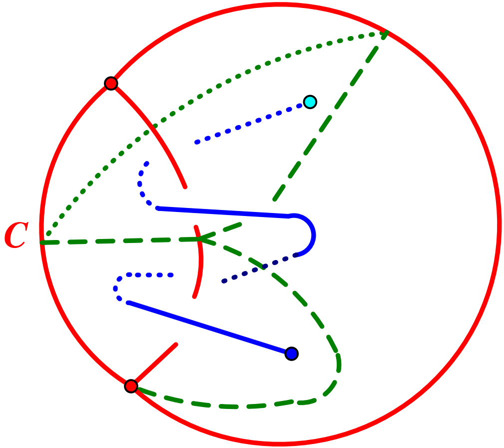
Proof sketch (assuming G is a $3-$connected graph with a flat spatial embedding):

Let P be a (maximally) planar subgraph of G, where G is embedded flatly. Let S be a sphere that contains P, as guaranteed by Wu. Choose S so that it intersects $G - P$ transversely, and so that it has a minimal number of intersections with $G - P$. If there are no intersections, we are done.

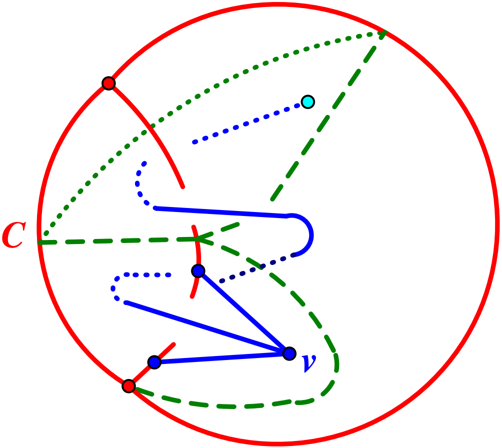
Otherwise, let C be the boundary of a face of P in S that intersects $G - P$ (by 2-connectivity, C is a cycle).



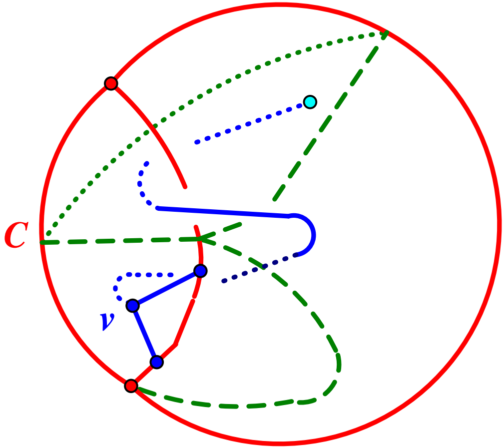
Take a panelling disk, D, for C that intersects S transversally and minimally. The intersection of C and S will be the union of C and circles and arcs with endpoints in C.



Take an endpoint of an edge whose interior intersects S (in the figure, vertex v). It will be contained in a component of $S - (C \cup D)$.



The portion of P that lies in this component can be ambient isotoped so that it lies in the face of C. The result is an embedding of P on S that has fewer intersections with $G - P$. This is a contradiction of the minimality of S.



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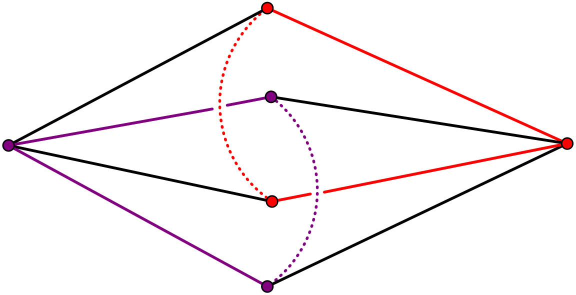
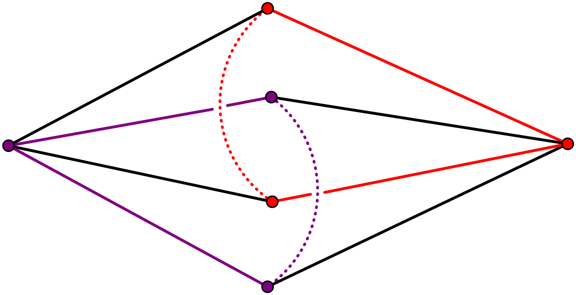
In particular, they determine a sphere, with fragments either lying inside or outside the sphere.

Conflict

Let G be a graph with M a maximally planar subgraph of G. Two $M-$ fragments, F and F' **conflict** if the vertices of attachment of F, v_1 and v_2 and the vertices of attachment of F', w_1 and w_2, form the 4-partition of a $K_{4,2}$ subgraph [or expansion of $K_{4,2}$] of M, such that, for every S^2 embedding of M, v_1 and v_2 do not lie in the same face of the induced embedding of the $K_{4,2}$.



One can use Conway-Gordon and Sach's proof that K_6 is intrinsically linked to show that every embedding of G with maximal planar subgraph M and with conflicting $M-$fragments F and F' on the same side of $M \subset S^2$, will contain a pair of linked cycles, and thus not be a flat embedding.



Conjecture: Give a maximal planar subgraph M, with exactly two $M-$fragments (edges) F and F', if F and F' do not conflict, then there exists a flat embedding of $M \cup F \cup F'$ with F and F' on the same side of $M \subset S^2$ in the flat embedding.

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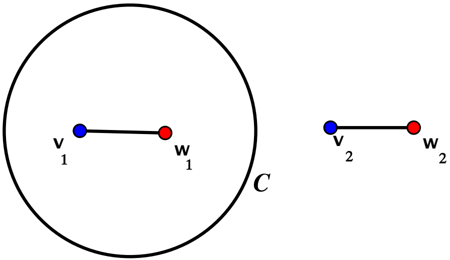
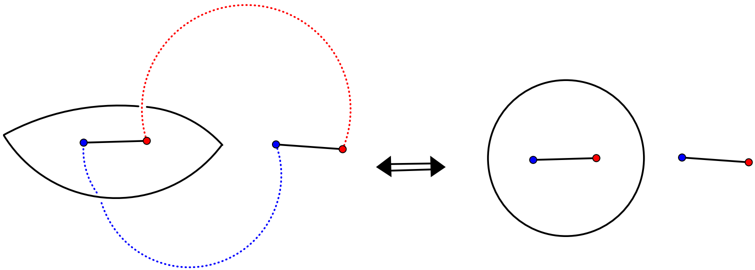
Seems very fundamental and deep.

Let G be a graph with M a maximally planar subgraph. Let F and F' be two fragments of M. We say that F and F' **anti-conflict** if for every S^2 embedding of M, there exists a cycle C in M,with the vertices of attachment of F (F') lying in different components of $S^2 - C$, and there is a path (possibly trivial) in M, connecting v_1 and w_1 in $S^2 - C$ and a path (possibly trivial) in M connecting v_2 and w_2 in $S^2 - C$.





Given a graph G and a maximally planar subgraph M, we form the (signed) conflict graph associated to M by associating each fragment of M to a vertex, and connecting vertices that correspond to **conflicting** fragments by a **negative** edge, and vertices that correspond to *anti-conflicting* fragments by a *positive* edge.

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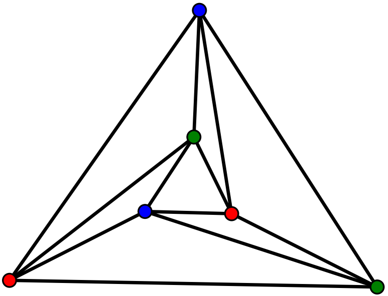
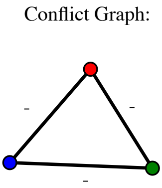
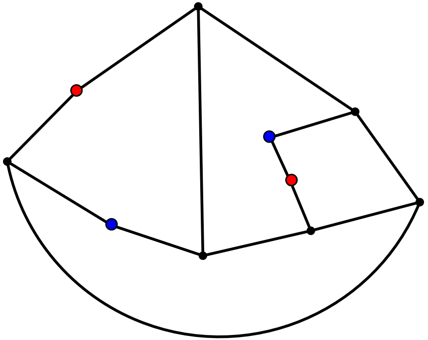
The forward direction is automatic. The converse is more difficult.

Evidence of the conjecture: the Petersen Family Graphs (proven by Robertson, Seymour and Thomas to form the complete minor-minimal set of graphs that do not have a flat embedding). A maximal planar subgraph of K_6 and its associated conflict graph:

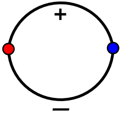




A maximal planar subgraph of the classic Petersen graph and its associated conflict graph:

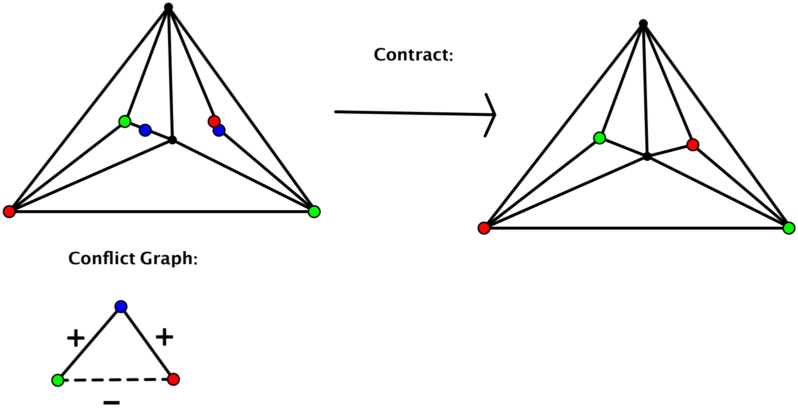


Associated conflict graph:



What can go wrong?

A maximally planar subgraph of P_7 with two fragments that need the third to "conflict".



New and improved definition:

Given a graph G with maximal planar subgraph M and fragments F and F', we say F and F' **weakly (anti-)conflict** if there exist a (possibly empty) list of edges in M, $e_1, e_2, ..., e_n$, such that contracting G along these edges results in $H = G/e_1/e_2/.../e_n$ and $M' = M/e_1/e_2/...e_n$ with $M' \subseteq M''$, where M'' is maximal planar in H, and F and F' are (anti-)conflicting fragments of M''.

Form the weak conflict graph in the usual way.

Now we can finish the proof that a graph G has a flat spatial embedding if and only if every maximal planar subgraph of G has a balanced weak conflict graph.

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So assume that G does not have a flat spatial embedding. By RST, G contains a Petersen Family graph, P, as a minor. To obtain P, first remove edges and vertices from G, to obtain G', and do all edge contractions last to go from G' to P.

The collection of edges used to contract G' down to P does not contain any cycles. The collection of such edges is thus contained in a maximal planar subgraph M' of G'.

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This was done by my REU students, but needs to be checked.

Final observation: it would be excellent to show directly (that is, without using RST) that if G has a maximal planar subgraph M with a balanced weak conflict graph, then G has a flat embedding.

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First step would be to prove:

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Thanks to Garry Bowlin (former REU, did doctoral thesis on signed graphs, now employed by Epic).