Reduced Wu and Generalized Simon Invariants

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June 9, 2013

Spatial Graphs Conference

There are many spatial graph invariants, but all have limitations.

Some examples are:

- Yamada polynomial – ambient isotopy invariant for 3-valent graphs, otherwise it's only a regular isotopy invariant (i.e., it isn't invariant under Reidemeister 1 moves).
- Yokota polynomial – ambient isotopy invariant, but difficult to compute, and can't distinguish mirror images.
- Wu invariant – homology invariant (hence ambient isotopy invariant), depends on labeling of vertices, and tedious to compute for a new graph.
- Simon invariant – ambient isotopy invariant, easy to compute, only defined for K_5 and $K_{3,3}$, depends on labeling of vertices.

Wu and Simon invariants depend on vertex labels

The Wu or Simon invariant can be used to show:

For any embedding Γ of K_5 or $K_{3,3}$ in S^3, there is no orientation reversing homeomorphism of (S^3, Γ) which fixes every vertex.

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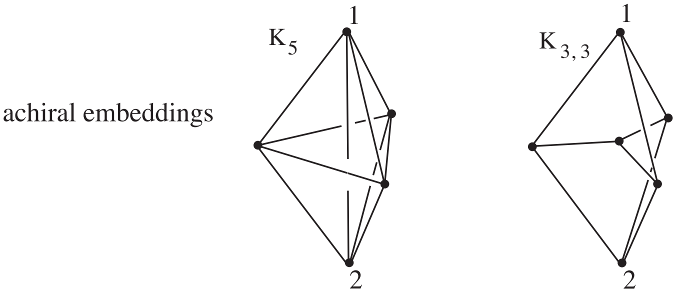
However, these graphs have a achiral embeddings if you don't require vertices to be fixed.



Reflection interchanges vertices 1 and 2.

Definition

A graph is said to be **intrinsically chiral** if no embedding of it in S^3 has an orientation reversing homeomorphism.

In this talk we define numerical invariants of spatial graphs with the properties:

- They are easy to compute.
- They can be used to prove intrinsic chirality.
- They give lower bounds for the minimum crossing number of an embedding.

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Let G have vertices $v_1, v_2, \ldots, v_m$ and oriented edges $e_1, e_2, \ldots, e_n$.

For every disjoint pair e_i and e_j define a variable $E^{e_i,e_j} = E^{e_j,e_i}$.

Let $Z(G)$ be the free $\mathbb{Z}$-module generated by the E^{e_i,e_j}'s.

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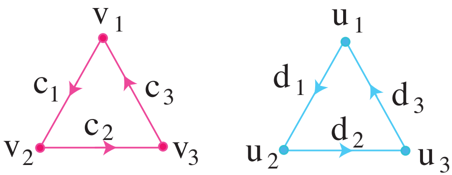
Example:
$2K_3$



$Z(2K_3) =$

$\langle E^{c_1,d_1}, E^{c_1,d_2}, E^{c_1,d_3}, E^{c_2,d_1}, E^{c_2,d_2}, E^{c_2,d_3}, E^{c_3,d_1}, E^{c_3,d_2}, E^{c_3,d_3} \rangle$

Taniyama's method for computing the Wu invariant

Write $I(k) = s$ and $T(k) = r$ to mean the oriented edge e_k has initial vertex v_s and terminal vertex v_r.

For every edge e_i and disjoint vertex v_s, define a variable V^{e_i,v_s}.

Definition

For a given edge e_i and disjoint vertex v_s, define

$$\delta(V^{e_i,v_s}) = \sum_{\substack{I(k)=s \\ e_i \cap e_k = \emptyset}} E^{e_i,e_k} - \sum_{\substack{T(j)=s \\ e_i \cap e_j = \emptyset}} E^{e_i,e_j} \in Z(G)$$

That is, $\delta(V^{e_i,v_s})$ is the sum of all edge variables disjoint from e_i with initial vertex v_s, minus the sum of all edge variables disjoint from e_i with terminal vertex v_s.

Taniyama's method for computing the Wu invariant

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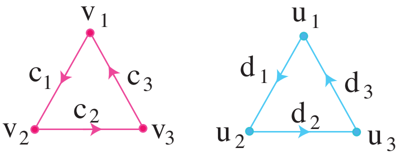

$$\delta(V^{c_1,u_1}) = \sum_{\substack{I(k)=u_1 \\ c_1\cap e_k=\emptyset}} E^{c_1,e_k} - \sum_{\substack{T(j)=u_1 \\ c_1\cap e_j=\emptyset}} E^{c_1,e_j} = E^{c_1,d_1} - E^{c_1,d_3}$$

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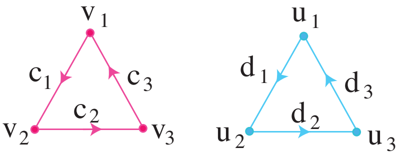

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Definition

$B(G)$ is defined as the submodule generated by all the $\delta(V^{e_i,v_s})$, and the **linking module** is defined as $L(G) = Z(G)/B(G)$.

In $L(K_{3,3})$, we have $[E^{c_1,d_1}] = [E^{c_1,d_3}]$

It can be shown that $L(2K_3) = \langle [E^{c_1,d_1}] \rangle \cong \mathbb{Z}$.

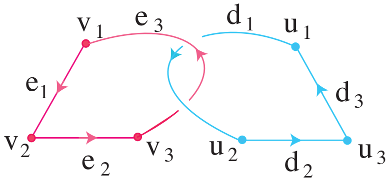
Taniyama's method for computing the Wu invariant

For an embedding $f : G \to S^3$, define $\ell(f(e_i), f(e_j))$ to be the sum of the signs of crossings between $f(e_i)$ and $f(e_j)$.

The Wu invariant is defined as

$$\mathcal{L}(f) = \sum_{e_i \cap e_j = \emptyset} \ell(f(e_i), f(e_j))[E^{e_i, e_j}] \in L(G)$$

Example



Recall $L(2K_3) = \langle [E^{e_1, d_1}] \rangle$.

$\mathcal{L}(f) = \sum \ell(f(e_i), f(d_j))[E^{e_1, d_1}] = 2\mathrm{lk}(f)[E^{e_1, d_1}] \in \langle [E^{e_1, d_1}] \rangle$

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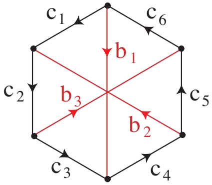
We obtain an integer valued invariant as follows.

Let G be a labeled graph with oriented edges and $\varepsilon : L(G) \to \mathbb{Z}$ be a homomorphism. For any embedding f of G, define the **reduced Wu invariant** $\tilde{\mathcal{L}}_\varepsilon(f)$ as the integer $\varepsilon(\mathcal{L}(f))$. We write $\varepsilon(e_i, e_j)$ for $\varepsilon([E^{e_i,e_j}])$, then $\tilde{\mathcal{L}}_\varepsilon(f) =$

$$\varepsilon\left(\sum_{e_i \cap e_j = \emptyset} \ell(f(e_i), f(e_j))[E^{e_i,e_j}]\right) = \sum_{e_i \cap e_j = \emptyset} \ell(f(e_i), f(e_j))\varepsilon(e_i, e_j)$$

That is, the sum of the crossing numbers between disjoint pairs of edges multiplied by integer coefficients.

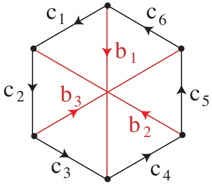
$K_{3,3}$



It can be shown (this is tedious) that for every pair of disjoint edges a and b, in $L(K_{3,3})$ we have $[E^{a,b}] = \varepsilon(a,b)[E^{c_1,c_3}]$ where $\varepsilon(c_i, c_j) = 1$, $\varepsilon(b_i, b_j) = 1$, and

$$\varepsilon(c_i, b_j) = \begin{cases} 1 & \text{if } c_i \text{ and } b_j \text{ are parallel} \\ -1 & \text{if } c_i \text{ and } b_j \text{ are anti-parallel} \end{cases}$$

$K_{3,3}$



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Thus $L(K_{3,3}) = \langle [E^{c_1,c_3}] \rangle$. Hence for any embedding f of $K_{3,3}$ the Wu invariant of f is

$$\mathcal{L}(f) = \sum \varepsilon(a,b)\ell(f(a), f(b))[E^{c_1,c_3}]$$

The reduced Wu invariant of an oriented $K_{3,3}$

From previous slide the Wu invariant of $f : K_{3,3} \to S^3$ is

$$\mathcal{L}(f) = \sum_{a \cap b = \emptyset} \varepsilon(a,b)\ell(f(a), f(b))[E^{c_1,c_3}]$$

where $\varepsilon(a,b)$ is $\varepsilon(c_i, c_j) = 1$, $\varepsilon(b_i, b_j) = 1$, and

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We define $\varepsilon : L(K_{3,3}) \to \mathbb{Z}$ by giving the value of ε on the generator $[E^{c_1,c_3}]$ as $\varepsilon(c_1, c_3) = 1$.

Thus the reduced Wu invariant for $f : K_{3,3} \to S^3$ is the integer

$$\tilde{\mathcal{L}}_\varepsilon(f) = \sum_{a \cap b = \emptyset} \varepsilon(a,b)\ell(f(a), f(b))$$

Reduced Wu invariant for an embedding of oriented $K_{3,3}$

Recall, $\varepsilon(a, b)$ is $\varepsilon(c_i, c_j) = 1$, $\varepsilon(b_i, b_j) = 1$, and

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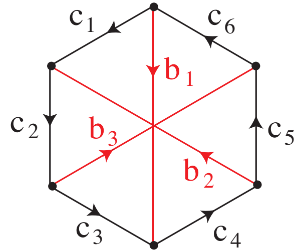
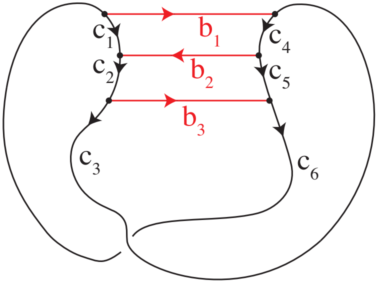


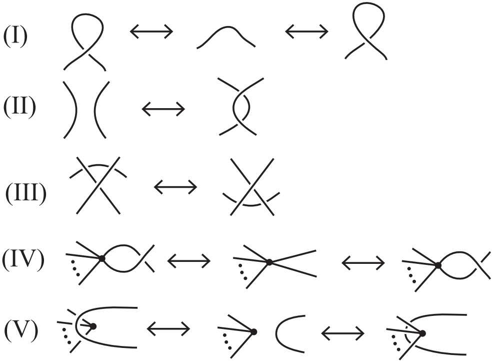


So for this embedding

$\tilde{\mathcal{L}}_\varepsilon(f) = \sum_{a \cap b = \emptyset} \varepsilon(a, b)\ell(f(a), f(b)) = \varepsilon(c_3, c_6) \times (-1) = 1$

For the graphs $K_{3,3}$ and K_5, the Simon invariant is the same as the reduced Wu invariant.

However, Simon proved invariance up to isotopy directly by showing the value of $\sum_{a \cap b = \emptyset} \varepsilon(a,b)\ell(f(a), f(b))$ is unchanged by Reidemeister moves for spatial graphs.



By using Simon's method we can create isotopy invariants of other spatial graphs. In particular,

Let G be an oriented labeled graph. If we can define an integer-valued function $\varepsilon(a, b)$ such that for any projection of an embedding $f : G \to S^3$ the value of

$$\widehat{L}_\varepsilon(f) = \sum_{a \cap b = \emptyset} \varepsilon(a, b)\ell(f(a), f(b))$$

is invariant under the Reidemeister moves, then we say $\widehat{L}_\varepsilon$ is a **generalized Simon invariant** for G.

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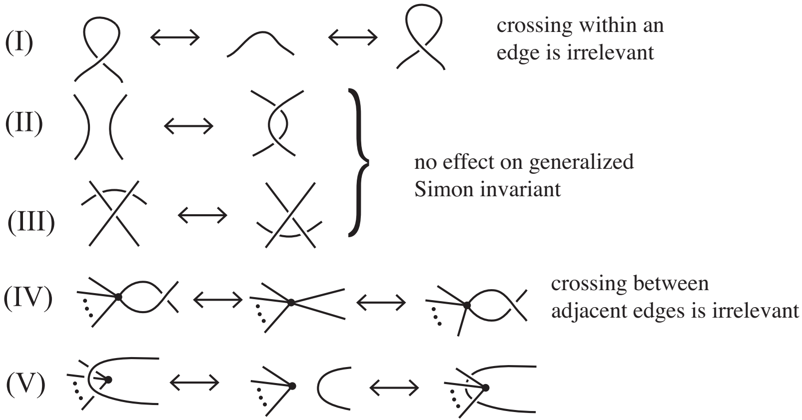
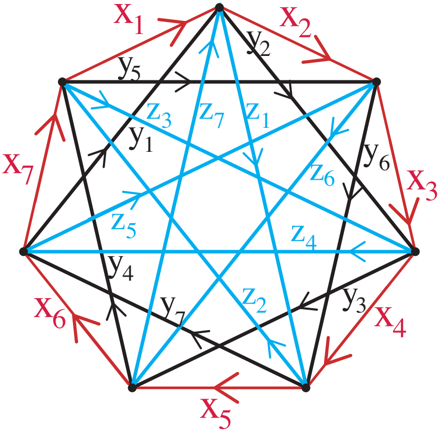
Since the reduced Wu invariant is a homology invariant, it is invariant under the Reidemeister moves.

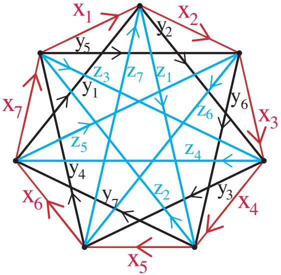
Thus any reduced Wu invariant is also a generalized Simon invariant.

The only difficulty in defining a generalized Simon invariant is proving that $\sum_{a \cap b = \emptyset} \varepsilon(a,b)\ell(f(a), f(b))$ is invariant under Reidmeister move (V).



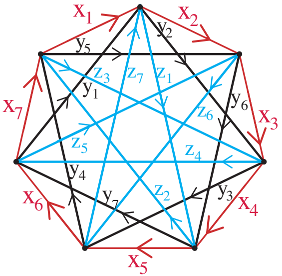


The oriented cycle $\overline{x_1 x_2 \ldots x_7}$ determines the oriented cycles $\overline{y_1 y_2 \ldots y_7}$ and $\overline{z_1 z_2 \ldots z_7}$.



Define $\varepsilon(x_i, y_j) = -1$ and

$$\varepsilon(x_i, x_j) = \varepsilon(y_i, y_j) = \varepsilon(z_i, z_j) = \varepsilon(x_i, z_j) = \varepsilon(y_i, z_j) = 1$$



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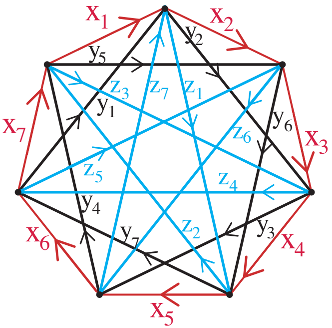
For any embedding f of K_7,

$$\widehat{L}_\varepsilon(f) = \sum_{a \cap b = \emptyset} \varepsilon(a, b)\ell(f(a), f(b))$$

is invariant under the Reidmeister moves.

Thus $\widehat{L}_\varepsilon(f)$ is a generalized Simon invariant.



Note that if we reverse the orientation of $\overline{x_1 x_2 \dots x_7}$, it reverses the orientations of all edges.

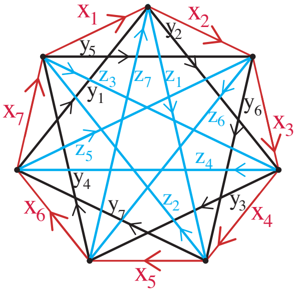
Hence $\widehat{L}_\varepsilon(f)$ does not depend on the orientation of $\overline{x_1 x_2 \dots x_7}$.

For every embedding of K_7, $\widehat{L}_\varepsilon(f)$ is odd

Lemma

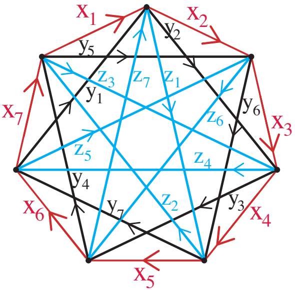
For any embedding f of K_7, the value of $\widehat{L}_\varepsilon(f)$ is odd.

Proof: Consider an embedding f with this projection, with some over-under information.



Each crossing has $\varepsilon(a, b) = 1$, since there are no crossings between any x_i and any y_j.

For every embedding of K_7, $\widehat{L}_\varepsilon(f)$ is odd



There are 35 crossings and all $\varepsilon(a, b) = 1$.

Thus $\widehat{L}_\varepsilon(f) = \sum_{a \cap b = \emptyset} \varepsilon(a, b)\ell(f(a), f(b))$ is odd.

Any crossing change will change the signed crossing number between two edges by ± 2.

So for any embedding $\widehat{L}_\varepsilon(f)$ is odd.

Proof that K_7 is intrinsically chiral

Theorem

K_7 is intrinsically chiral.

Proof: Suppose that K_7 has an achiral embedding f.

Then there is an orientation reversing homeomorphism $h : (S^3, f(K_7)) \to (S^3, f(K_7))$.

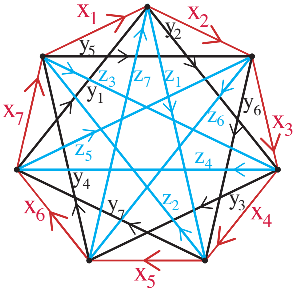
Let J denote the set of Hamiltonian cycles with nonzero arf invariant.

By Conway-Gordon, $|J|$ is odd.

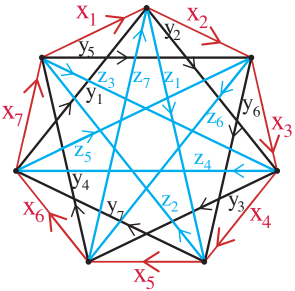
h permutes the elements of J, so the order of some orbit is an odd number n.

$\therefore$ h^n setwise fixes some Hamiltonian cycle with nonzero arf invariant.

Proof that K_7 is intrinsically chiral



Let $\overline{x_1 x_2 \ldots x_7}$ be a Hamiltonian cycle preserved by h^n. This defines the oriented cycles $\overline{y_1 y_2 \ldots y_7}$ and $\overline{z_1 z_2 \ldots z_7}$.



Let $\overline{x_1 x_2 \ldots x_7}$ be a Hamiltonian cycle preserved by h^n. This defines the oriented cycles $\overline{y_1 y_2 \ldots y_7}$ and $\overline{z_1 z_2 \ldots z_7}$.

Since $\overline{x_1 x_2 \ldots x_7}$ is setwise invariant under h^n, the cycles $\overline{y_1 y_2 \ldots y_7}$ and $\overline{z_1 z_2 \ldots z_7}$ are also setwise invariant under h^n.

Thus h^n preserves all values of $\varepsilon(a, b)$.

Also, if h^n reverses the orientation of $\overline{x_1 x_2 \ldots x_7}$, then h^n reverses the orientation of all edges.

Since h^n is orientation reversing, for each pair of edges a and b

$$\ell(h^n(f(a)), h^n(f(b))) = -\ell(f(a), f(b))$$

Since h^n preserves all values of $\varepsilon(a, b)$, this means

$$\widehat{L}_\varepsilon(h^n \circ f)) = -\sum_{a \cap b = \emptyset} \varepsilon(a, b)\ell(f(a), f(b)) = -\widehat{L}_\varepsilon(f)$$

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But since $h^n(f(K_7)) = f(K_7)$, we also have

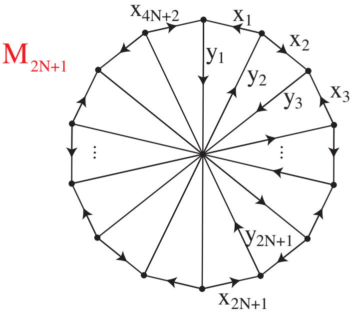
$$\widehat{L}_\varepsilon(h^n \circ f)) = \widehat{L}_\varepsilon(f)$$

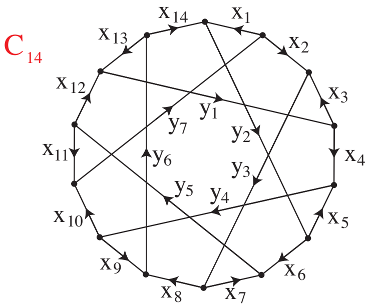
Thus $\widehat{L}_\varepsilon(f) = -\widehat{L}_\varepsilon(f)$. $\Longrightarrow\Longleftarrow$ since $\widehat{L}_\varepsilon(f)$ is odd.

Therefore K_7 is intrinsically chiral.

Other Examples

We use generalized Simon invariants to prove Möbius ladders M_{2N+1} with $N > 1$ and Heawood graph C_{14} are intrinsically chiral.





- Every homeomorphism of $(S^3, f(M_{2N+1}))$ with $N > 1$ leaves the cycle $\overline{x_1 \ldots x_{4N+2}}$ setwise invariant [Simon].
- Every homeomorphism of $(S^3, f(C_{14}))$leaves either a 14-cycle or a 12-cycle setwise invariant [Nikkuni].

Generalized Simon invariants can be used to show that a particular projection of a spatial graph has minimal crossing number.

Generalized Simon invariants can be used to show that a particular projection of a spatial graph has minimal crossing number.

Theorem

Let f be an embedding of an oriented graph G in S^3 with generalized Simon invariant $\widehat{\mathcal{L}}_\varepsilon(f)$, and let $c(f)$ be the minimal crossing number of f. For a given projection of $f(G)$, let $m_\varepsilon(f)$ be the maximum of $|\varepsilon(e_i, e_j)|$ over all disjoint edges with $\ell(f(e_i), f(e_j)) \neq 0$. Then

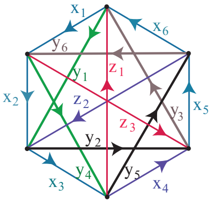
$$c(f) \geq \frac{|\widehat{\mathcal{L}}_\varepsilon(f)|}{m_\varepsilon(f)}$$

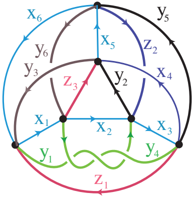
Proof:

$$\begin{aligned}|\widehat{\mathcal{L}}_\varepsilon(f)| &= \left|\sum_{e_i\cap e_j=\emptyset} \varepsilon(e_i,e_j)\ell(f(e_i),f(e_j))\right| \\ &\leq \sum_{e_i\cap e_j=\emptyset} |\varepsilon(e_i,e_j)||\ell(f(e_i),f(e_j))| \\ &\leq m_\varepsilon(f)\sum_{e_i\cap e_j=\emptyset} |\ell(f(e_i),f(e_j))| \\ &\leq m_\varepsilon(f)c(f)\end{aligned}$$

Thus

$$c(f) \geq \frac{|\widehat{\mathcal{L}}_\varepsilon(f)|}{m_\varepsilon(f)}$$



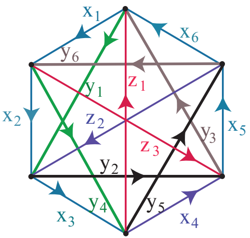


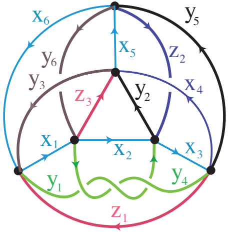
$\varepsilon(z_i, z_j) = 1$, $\varepsilon(x_i, y_j) = -1$, $\varepsilon(y_i, z_j) = 0$

$$\varepsilon(x_i, x_j) = \begin{cases} 3 & \text{if } x_i \text{ and } x_j \text{ are anti-parallel} \\ 2 & \text{if } x_i \text{ and } x_j \text{ are neither parallel nor anti-parallel} \end{cases}$$

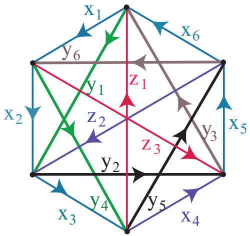
$$\varepsilon(y_i, y_j) = \begin{cases} 0 & \text{if } y_i \text{ and } y_j \text{ are anti-parallel} \\ -1 & \text{if } y_i \text{ and } y_j \text{ are neither parallel nor anti-parallel} \end{cases}$$

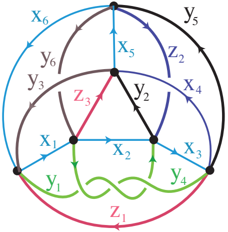
$$\varepsilon(x_i, z_j) = \begin{cases} -1 & \text{if } x_i \text{ and } z_j \text{ are anti-parallel} \\ 1 & \text{if } x_i \text{ and } z_j \text{ are parallel} \end{cases}$$





$m_\varepsilon(f)$ is maximum of $|\varepsilon(e_i, e_j)|$ over all disjoint edges with $\ell(f(e_i), f(e_j)) \neq 0$. So $m_\varepsilon(f) = 1$.





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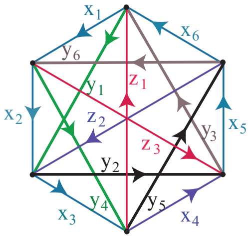
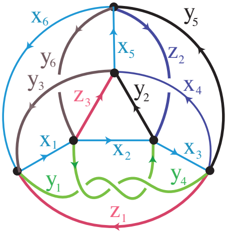
$$\tilde{\mathcal{L}}_\varepsilon(f) = \varepsilon(y_3, y_6)\cdot 1 + \varepsilon(x_4, z_2)\cdot 1 + \varepsilon(y_1, y_4)\cdot 3 = -1 - 1 - 3 = -5$$





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By the theorem,

$$c(f) \geq \frac{|\widehat{\mathcal{L}}_\varepsilon(f)|}{m_\varepsilon(f)} = 5$$

Thus this projection has minimal crossing number.

Thanks

Thanks for listening and for coming to our conference.

See you in Tokyo in August.