

Three-dimensional finite point groups and the symmetry of beaded beads

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Beaded beads are clusters of beads woven together (usually around one or more large holes). Their groups of symmetries are classified by the three-dimensional finite point groups, i.e. the finite subgroups of the orthogonal group of degree three, $O(3)$. The question we answer is whether *every* finite subgroup of $O(3)$ can be realized as the group of symmetries of a beaded bead. We show that this is possible, and we describe general weaving techniques we used to accomplish this feat, as well as examples of a beaded bead realizing each finite subgroup of $O(3)$ or, in the case of the seven infinite classes of finite subgroups, at least one representative beaded bead for each class.

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Section 1: Introduction

Weavers of beads use a needle and thread to sew beads together to make decorative objects including jewelry, wall hangings, and baskets. Many weavers of beads iterate patterns to create designs for cables and straps for necklaces, bracelets, and hand bags. Bead weave designers such as Diane Fitzgerald, Sharri Moroshok, Christine Prussing, Takako Samejima, Laura Shea, Florence Turnour, and Carol Wilcox Wells weave beads into composite clusters, usually with at least one large hole, called *beaded beads* [1-7]. This style of beading has proved to be very popular and is spreading rapidly. Mathematically, many beaded beads can be viewed as polyhedra, with each bead (or, more precisely, the hole through the middle of each bead, which provides its orientation) corresponding to an edge of the polyhedron. Different weaving patterns will bring different numbers of these “edges” together to form the vertices of the polyhedron. So it is very natural to use various polyhedra as the inspiration for beaded bead designs.

Common beaded bead designs include the one based upon the cube shown in figure 1. The holes of the 12 beads lie upon the 12 edges of the cube. The photo on the left shows the view of the cube looking at an edge, the photo in the middle shows the view looking at a vertex, and the photo on the right shows the view looking at a face.



Figure 1: A Beaded Cube

Another well-known pattern is the *beaded ball*, which weaves 30 beads into a dodecahedron. Views looking at an edge, vertex and face (from left to right) are shown in figure 2.



Figure 2: A Beaded Ball

Both of these designs are hollow, held open by the tension of the thread and the packing of the beads. All five of the regular polyhedra can be sewn in this manner. Our goal is to show, in a sense, that *any* polyhedron can be modeled as a beaded bead – more specifically, we will show that, given any polyhedron, we can weave a beaded bead with the same set of symmetries. The challenge is to create the patterns to accomplish this so that we are also creating beaded beads that are objects of beauty in their own right. In meeting this challenge, we developed many new designs that may not have been created otherwise.

In this paper we will review the possible groups of symmetries of polyhedra (otherwise known as the finite three-dimensional point groups), and will demonstrate how to create beaded bead patterns to realize each of them. All of the beaded beads presented in this paper measure between 12mm and 38mm, but the designs could be scaled to virtually any size.

Section 2: The symmetry groups of polyhedra, and their realizations via beaded beads

A *symmetry* of a geometric object is a rigid transformation of space that leaves the appearance of the object unchanged. We are considering bounded objects in three dimensions, so we only consider rigid transformations that fix at least one point (thus ruling out translations). Any of these transformations is a rotation around an axis through the origin (the fixed point), a reflection across a plane through the origin, or a roto-reflection (a rotation around the axis followed by a reflection across a plane through the origin). Since combining two symmetries of an object results in another symmetry, the collection of symmetries for any object is a group under composition, called the *symmetry group* of that object.

2.1 Symmetries of a sphere

Any rotation, reflection, or roto-reflection that fixes the origin also fixes a sphere centered at the origin. The symmetry group of a sphere is the (infinite) collection of all these motions (commonly denoted $O(3)$). Any given polyhedron will share some finite collection of these symmetries, so its symmetry group is a finite subgroup of $O(3)$. There are infinitely many such subgroups, but they naturally divide into 7 infinite classes of groups (the *prismatic groups*, corresponding to the 7 infinite frieze groups) and 7 additional groups (corresponding to the symmetry groups of the Platonic solids and their subgroups) [8, 9].

2.2 The prismatic groups

An *n-sided right regular prism* is a polyhedron consisting of two regular *n*-sided polygons joined by rectangles. For example, the box for a Toblerone chocolate bar is a 3-sided right regular prism, and a pencil is a 6-sided right regular prism (ignoring the point and eraser, of course). The prismatic groups are the symmetry groups of right regular prisms and their subgroups – these are collected into 7 infinite classes indexed by *n*, the number of sides (excluding bases) of the corresponding prisms. Each of these groups contains a rotation symmetry (of order *n*) around an axis through the center of the prism (like the lead in a pencil). In addition, there may be reflection symmetries through planes through the axis of rotation or perpendicular to it, and/or rotations (of order 2) around lines perpendicular to the main axis of the prism.

The seven classes of prismatic groups correspond to the seven infinite border (i.e. frieze) groups. To see the association, simply take a strip of three or more translational repeats from a border pattern, then wrap this strip around a prism so that one repeat is placed on each side, and you have an object with the associated prismatic group. Rotations around the central axis in the prismatic group correspond to translations in the frieze group. Accordingly, for each of the seven frieze groups, the associated prismatic groups form a countably infinite class of groups, one for each number of repeats/sides. We will look at each family of prismatic groups in turn, and provide examples of at least one beaded bead one of the symmetry groups in each class; in what follows, *n* always represents the order of the rotation symmetry around the central axis. For *n* equal to one or two, the prisms are degenerate, and some of the families of prismatic groups overlap [8, 10]. This distinction is not relevant for bead construction so is not addressed further in this paper.

Recall that the seven frieze groups are typically denoted 11 (translation only), $1m$ (translation and horizontal reflection), $m1$ (translation and vertical reflection), 12 (translation and 180° rotation), $1g$ (translation and glide reflection), mg (translation, horizontal reflection and glide reflection), mm (translation, horizontal and vertical reflections, and 180° rotation).

2.2.1 C_n . The first prismatic group we consider corresponds to the frieze pattern 11 . The footprint pattern in figure 3 shows 11 symmetry.



Figure 3: 11 frieze pattern

When transferred to an n -sided prism, the translation symmetry of the frieze pattern corresponds to the n -fold rotation symmetry in C_n . Erin Simonetti's beaded bead in figure 4 exhibits C_3 symmetry (not including the bail).



Figure 4: C_3 prismatic group

2.2.2 C_{nh} . The next prismatic group corresponds to the $1m$ frieze pattern, shown in figure 5.



Figure 5: Im frieze pattern

The corresponding prismatic group C_{nh} has both a central rotation symmetry and a reflection symmetry through a “horizontal” plane perpendicular to the axis. The photos in figure 6 show the top and side views of a C_{5h} symmetric beaded bead made using the Fringe Method described at the end of this section on prismatic groups. Beaded beads with this symmetry group are very uncommon, so rather than using a known pattern the authors deliberately created this design to realize the C_{5h} symmetry group.



Figure 6: C_{5h} prismatic group

2.2.3 C_{nv} (pyramidal symmetry). The symmetry of an n -sided right regular pyramid corresponds to the frieze group $m1$, shown in figure 7.



Figure 7: $m1$ frieze pattern

The corresponding symmetry group C_{nv} has a set of n “vertical” mirror planes, each containing the axis of rotation. This is a very common symmetry group for beaded beads. Figure 8 shows two photos of a ‘Seven Sisters’ beaded bead with C_{7v} symmetry, and another beaded bead using the same technique with C_{3v} symmetry.



Figure 8: C_{7v} and C_{3v} prismatic groups

2.2.4 D_n (dihedral group). The dihedral group corresponds to the frieze group $I2$, shown in figure 9.

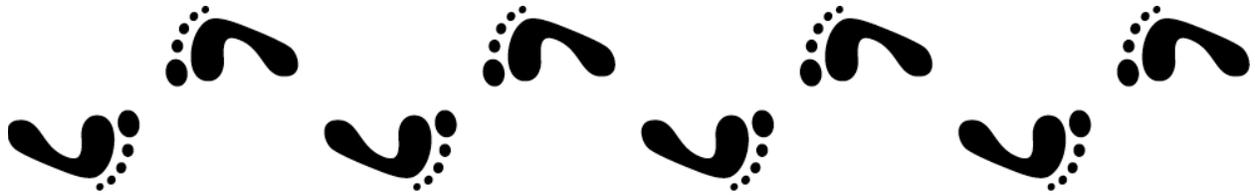


Figure 9: $I2$ frieze pattern

D_n has rotation axes of order two that are perpendicular to the main axis of rotation, and no planes of reflection. The ‘Embellished Daisy Bead’ shown in figure 10 exhibits D_5 symmetry and uses the well-known daisy chain stitch together with the Fringe Method described later.



Figure 10: D_5 prismatic group

2.2.5 S_{2n} . The next prismatic group, S_{2n} (not to be confused with the group of permutations of $2n$ objects), corresponds to the frieze group Ig , shown in figure 11.



Figure 11: $1g$ frieze pattern

S_{2n} has a roto-reflection symmetry combining a rotation (by half as much as the central rotation symmetry) around the central axis with a reflection through a plane perpendicular to the axis. figure 12 shows a bead with symmetry group S_8 – notice that the central rotation symmetry has order 4, while the roto-reflection symmetry has order 8. The object in figure 12 appears to have vertical mirror reflections but one should notice that the little beads are dark blue on only one side of the big blue beads.



Figure 12: S_8 prismatic group

2.2.6 D_{nd} (antiprismatic symmetry). Antiprismatic symmetry corresponds to the frieze pattern mg , shown in figure 13.



Figure 13: mg frieze pattern

The symmetries of antiprisms include vertical planes of reflection through the main axis of rotation, but no plane of reflection perpendicular to the axis. Instead, there is a perpendicular plane of roto-reflection. The bead on the left of figure 14 exhibits D_{3d} symmetry in which the main axis of rotation is horizontal. The two photos to the right of figure 14 show the ‘Four of Spades’ beaded bead made with the Fringe Method. If this beaded bead were uncolored, it would exhibit D_{4d} symmetry.



Figure 14: Beaded beads representing the D_{3d} and D_{4d} prismatic groups

2.2.7 D_{nh} (the full prismatic group). The full prismatic group corresponds to the frieze group mm , shown in figure 15.

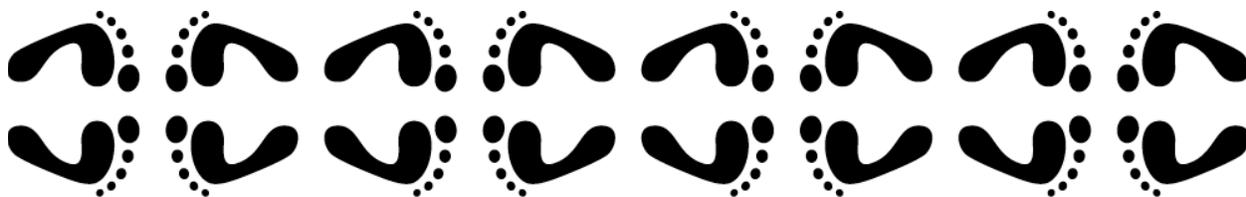


Figure 15: mm frieze pattern

The full prismatic group adds both a horizontal plane of reflection and n vertical planes of reflection to the central n -fold rotation symmetry. Each of the n lines of intersection of the horizontal plane of reflection with a vertical plane of reflection is an axis of rotation of order 2. The ‘Spinning Top’ beaded beads on the left of figure 16 have D_{10h} symmetry and are made with the Fringe Method described in the next section. The two photos on the right of figure 16 show the ‘Double Rose Window’ beaded bead with D_{6h} symmetry. Like C_{nv} , a wide variety of bead weaving techniques can be applied to make beaded beads with D_{nh} symmetry. Because it is so easy to make, this type of symmetry on beaded beads is perhaps the most common type.



Figure 16: D_{10h} and D_{6h} prismatic groups

2.2.8 A general method for weaving a prismatic beaded bead -- the Fringe Method

All of the prismatic symmetries can be exhibited in beaded beads using a technique developed by Florence Turnour and the first author, called the Fringe Method [6]. To make a ‘Spinning Top’ beaded bead (figure 16) using the Fringe Method, start by stringing a few beads with a needle and thread; the illustrations in figure 17 show seven beads. Make a strand of fringe by sewing back through beads 6, 5, 4, 3, and 2 again, in the opposite direction, as shown. Then, sew through bead 1 in the original direction, completing the strand of fringe. Create several more strands of fringe (ten total are shown), sliding each fringe up the thread so that it lies snugly against the previous strand of fringe. Arrange the fringe in a sun shape, so that bead 1 in the last strand of fringe is next to bead 1 of the first strand of fringe. Connect the circle, sewing through all ten beads, and then pass through bead 1 of the first strand of fringe again. Pull the thread tight enough that the beads in the circle fit snugly together. Sew up through beads 2 through 6 to the top of the first strand of fringe, and pass through the fringe tip (bead 7). Sew through the fringe tip in the next strand of fringe. Pull the thread through as far as you can without disturbing the sun shape. Continue around the sun shape, several beads past the first strand of fringe as shown in figure 17 (right).

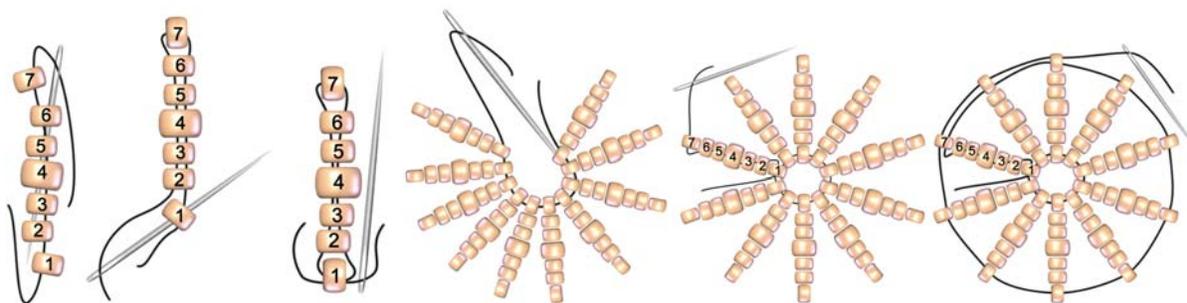


Figure 17: The Fringe Method

Place a large (core) bead in the middle of the sun shape, and hold it there with your index finger. Pull the thread slowly, so the fringe closes snugly around the large bead. Sew the rest of the way around the circle again, ending by passing through the top of the first strand of fringe. Tie off and cut both ends of your thread and the beaded bead is complete.

To make all of the prismatic group symmetries with the fringe method, we use more elaborate strands of fringe in the first steps rather than the plain fringe shown. When the length and number of the strands of fringe change, the size of the core bead needs to change accordingly for the beads to fit properly. Gaps between the pieces of fringe can be embellished with other beads. For specifications on bead sizes and other practical considerations, see [11].

2.3 Symmetry groups of the Platonic solids and their subgroups

The seven finite point groups that are not prismatic are subgroups of the groups of symmetries of the Platonic solids. In contrast with the prismatic groups, these groups each have more than one axis of 3-, 4-, or 5-fold symmetry. Three of the Platonic point groups are the full symmetry groups of the five Platonic solids: T_d is the symmetry group of the tetrahedron, O_h is the

symmetry group of both the cube and the octahedron, and I_h is the symmetry group of both the dodecahedron and the icosahedron. The remaining four point groups are proper subgroups of these. All seven are discussed below.

2.3.1 T_d (full tetrahedral symmetry group). A tetrahedron has four axes of rotational symmetry, each of order three. The full tetrahedral symmetry group, T_d , is of order 24 and is isomorphic to the symmetric group S_4 .

There are several ways to build a polyhedron with beads. One technique that will always work is to align the hole of a bead along each edge of the polyhedron as shown for the cube and dodecahedron in the introduction. Then, the thread connects the beads at the vertices of the polyhedron. In a tetrahedron, three beads meet at every vertex. Any tetrahedral beaded bead will naturally require six identical sets of beads, one for each of the six edges of the tetrahedron. Rather than give an example of the simplest tetrahedron, we have used a more complex design based on the structure resulting from the second iteration in the construction of the Sierpinski Tetrahedron, as shown in figure 18.

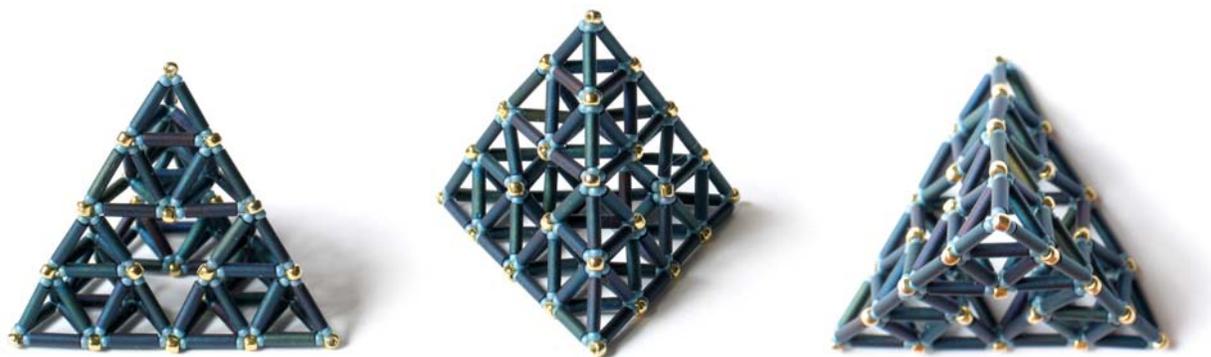


Figure 18: A beaded Sierpinski Tetrahedron

2.3.2 O_h (full octahedral symmetry group). Since the cube and the octahedron are dual, their symmetry groups are isomorphic. O_h has 48 symmetries, including rotation symmetries of orders four, three and two. There are three axes of rotation of order four, one through each pair of opposite faces of the cube. Through each of the four pairs of opposite vertices of a cube are axes of order three; and through each of the six pairs of opposite edges are axes of order two. Figure 1 shows the basic beaded cube, which has symmetry group O_h . Figure 19 shows another beaded bead with symmetry group O_h . The three photos show the view looking straight down along each of the three types of axes of rotation. The left photo shows an order three axis of rotation (the symmetries of an equilateral triangle), the middle photo shows an order four axis of rotation (the symmetries of a square), and the right photo shows an order two axis of rotation (the symmetries of a rectangle). The colors of the beaded bead subtly break some of the symmetry; the trios of beads at the vertices of the cube alternate between two color schemes. This beaded bead is hollow and has holes running through seven different axes of symmetry.



Figure 19: The full octahedral symmetry group O_h

Another polyhedron with symmetry group O_h is a cuboctahedron, which can be visualized as a cube with all of the corners cut off. Figures 20 and 21 show how the net of a cuboctahedron can be used to create a lattice of beads around a sphere or a core bead. The illustration on the left of figure 20 shows a bead weaving pattern where the smallest beads correspond to the edges of the cuboctahedron, and the largest beads to the vertices. A photograph of this pattern is shown in the middle photograph, and a central bead is added on the right.



Figure 20: Stages in beading a cuboctahedron

Finally, a circle of beads connects the top four vertex beads to complete the cuboctahedron beaded bead, as shown in figure 21. Additional smaller beads were added to the square faces for embellishment.



Figure 21: A finished cuboctahedron

2.3.3 I_h (full icosahedral symmetry group). The technique used to sew a cuboctahedron can be generalized to make other polyhedra, including the icosahedron shown on the left in figure 22. The full symmetry group of the icosahedron has axes of rotation of orders two, three, and five, corresponding to the edges, faces, and vertices, respectively, of the icosahedron. The dodecahedron, as the dual of the icosahedron, has the same symmetry group. Two images of another beaded bead with this symmetry group are shown at the center and right of figure 22. The left beaded bead uses the method described for the cuboctahedron, plus an extra layer of beads which is added to make each triangular join rigid, so that no core bead is needed to keep the hollow beadwork in shape. The cuboctahedron technique can also be used to destroy selective symmetries. Examples of this are shown in this next section.



Figure 22: The full icosahedral symmetry group I_h

2.3.4 T (chiral tetrahedral symmetry group). The chiral tetrahedral group T is the group of *rotation* symmetries of the tetrahedron, ignoring the reflection symmetries. We can view T as the symmetry group of a tetrahedron with each side decorated to break the reflection symmetries, as shown in figure 23.

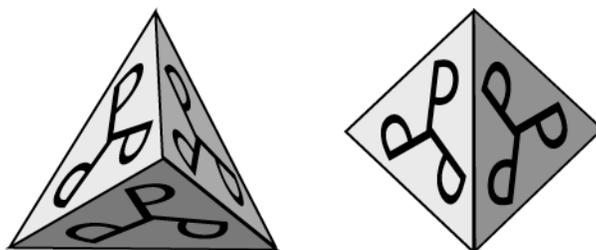


Figure 23: A labeled tetrahedron with symmetry group T

Beaded beads realizing symmetry group T (and other purely rotational symmetry groups) require the bead weaver to deliberately break the reflection symmetry. If a weaving pattern uses only one bead for each edge of a polyhedron, then there is no way to make the edges asymmetric. Designing a beaded bead with less symmetry requires the designer to start with a pattern with more redundancy, and these redundant beads can appear on the faces, edges, or vertices. Redundant beads can be identical in color or size, or they can differ, thereby destroying symmetry (i.e., reducing the symmetry group of the beaded bead to a subgroup of the full symmetry group of the underlying Platonic solid). Once a design is established, changing the colors of the redundant beads is relatively simple compared to changing the sizes (which will

affect how the beads fit together). With more beads in a design, the completed beaded bead is naturally larger. To keep the piece small, the challenge is to keep designs as simple as possible, while still exhibiting the desired symmetry. In the photos in figure 24, the beaded bead with full tetrahedral symmetry group T_d (on the right) is relatively simple compared to the beaded bead that exhibits only the chiral tetrahedral symmetry group T (left and centre).



Figure 24: The chiral tetrahedral symmetry group T as compared with T_d

2.3.5 O (chiral octahedral symmetry group). The symmetry group O is the subgroup of rotation symmetries of the full symmetry group of the cube (or octahedron). This and other chiral groups have no reflection symmetry. The beaded bead in figure 25 has symmetry group O . The left photo shows the view looking down along an axis of an order 3 rotation, and the right photo shows the order 4 rotation.



Figure 25: The chiral octahedral group O

2.3.6 I (chiral icosahedral symmetry group). Our third group of rotations is the group of rotations of the icosahedron. The beaded bead in figure 26 has symmetry group I and, like the bead in figure 25, was constructed using the same basic technique as the cuboctahedron shown in figure 21.



Figure 26: The chiral icosahedral group I

2.3.7 T_h (the pyritohedral symmetry group). The final symmetry group is another subgroup of O_h , consisting of the rotations of orders two and three and the reflections in planes parallel to the faces of the cube. This group is the *pyritohedral symmetry group* T_h . The illustration and the beaded bead in figure 27 both exhibit T_h symmetry. The beaded bead in this example is an example of the Borromean Rings (a link with three components, each pair of which is unlinked) and was sewn using a three-dimensional version of right angle weave called *box stitch*. The box stitch is a linear repeat of the cube beaded bead in figure 1, with additional beads added for embellishment and stiffness.

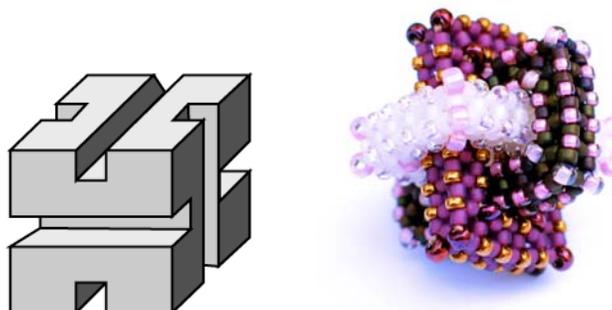


Figure 27: The pyritohedral symmetry group T_h

The beaded bead in figure 28 also exhibits T_h symmetry. This beaded bead was constructed using the same basic technique as for the cuboctahedron in figure 21.



Figure 28: A beaded bead with symmetry group T_h

Section 3: Conclusions and Future Work

We have demonstrated how any of the three-dimensional finite point groups can be represented in the medium of beads. Bead weaving is a phenomenally rich medium for creating mathematical art in three dimensions. For example, bead weavers also make decorated cables, surfaces, knots, tilings, networks, molecules, and Mobius-type bands, any of which can be colored in various ways. Also, the construction of beaded beads naturally leads to several mathematical questions. For example, what is the minimal number of beads required to realize each point group? What is the minimal length of thread required? We hope to investigate these and other questions in future work.

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