Notes for Math 198: Mathematics of Symmetry

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Contents

1 Introduction

This is a course on the *Mathematics of Symmetry*. What does that mean? How can symmetry have any mathematics to it? Before we can do anything else, we need to figure out what we mean by *symmetry*, and what we mean by *mathematics*.

Task 1. *You have been provided with a set of images. Which of these images are "symmetric"? Why? Discuss your answers with your classmates.*

According to *Webster's Ninth New Collegiate Dictionary*, *symmetry* is

1. beauty of form arising from balanced proportions
2. correspondence in size, shape, and relative position of parts on opposite sides of a dividing line or median plane or about a center or axis (see *bilateral symmetry*, *radial symmetry*)
3. a rigid motion of a geometric figure that determines a one-to-one mapping onto itself
4. the property of remaining invariant under certain changes (as of orientation in space, of the sign of the electric charge, of parity, or of the direction of time flow) – used of physical phenomena and of equations describing them

These definitions range from the aesthetic to the mathematical to the scientific, but there are common themes running through them. The idea of "balance" implies that there are different parts of the object which, in a sense, have the same "weight." From there, it is natural to think of the effect of manipulating the object so that these parts are interchanged, while still preserving some properties of the object.

We will be primarily studying the mathematical notion of symmetry – a rigid motion of a geometric figure which takes the figure to itself (i.e. which leaves it looking the same before and after). While the aesthetic notion of symmetry is much broader, we will see that objects that exhibit mathematical symmetry are frequently also aesthetically pleasing. In fact, these geometric symmetries are often important in the creation of art, and are particularly important in architecture, design and the decorative arts. As we study the mathematics of symmetry, we will also look at some of these applications to visual art and design.

This still leaves the question of what we mean by the mathematics of symmetry. According to *Webster's Ninth New Collegiate Dictionary*, *mathematics* is "the science of numbers and their operations, interrelations, combinations, generalizations, and abstractions, and of space configurations and their structure, measurement, transformations, and generalizations." In studying visual symmetry, we are going to be more interested in "space configurations" than in numbers – essentially, we are studying part of geometry. But even more important than the objects we study are the methods we use to study them, and this is what truly sets mathematics apart and makes it a powerful tool.

The process of mathematics involves recognizing patterns, making conjectures, and then proving (or disproving) your conjectures through rigorous logical analysis. This process the essence of mathematics – but it also includes skills that can be applied to problems and situations in any field of knowledge.

1.1 Goals of the Course

This course has several broad goals. One goal is to understand the mathematical notion of symmetry of visual patterns. Another is to see how these ideas can be applied in art and design. But the most important goal is to learn the process of doing mathematics. To learn this process, we need a problem to apply it to. In this course, we will be working on the following problem:

Final Goal. *Classify two-dimensional patterns and designs according to their symmetries.*

Of course, we can't even begin to work on this problem until we understand what it's asking. One of the most important lessons of mathematics is the importance of *definitions.* In casual conversation, and even many areas of academic discourse, it is common to use terms whose precise definitions are elusive, or dependent on context – in fact, this "fuzziness" of language is often used quite effectively in literature and poetry. In mathematics, however, this is anathema. It is vitally important to know *exactly* what is meant by any term in order to properly apply logical reasoning to it. The Final Goal stated above has three terms we need to define more precisely: "two-dimensional pattern", "symmetry" and "classify".

Definition 1. *A* two-dimensional pattern *is any set of points in the plane.*

So the term "pattern" here is slightly misleading – technically, *any* set of points in the plane, however asymmetric, is a "pattern". Of course, we will primarily be interested in sets that *do* exhibit some kind of symmetry, hence our use of "pattern". But this brings us again to what we mean by a *symmetry.*

Definition 2. *A* symmetry *of a two-dimensional pattern is a rigid motion of the plane which maps the pattern to itself.*

A *rigid motion* is something you do that doesn't stretch or compress the plane, so everything stays the same size (we'll give a more formal definition later). The idea of a symmetry is that you move the plane in some way, but someone who looked at the pattern before and after you do the motion wouldn't be able to tell that anything had changed. In the next section, we will explore this idea at more length.

Finally, what do we mean by *classify*? Classification problems arise in every field, not just in mathematics. Geologists classify rocks, biologists classify species of plants and animals, historians classify historical periods, experts on Shakespeare classify his plays, and so on. Classifications allow us to identify patterns among a range of objects, and use them to draw conclusions about individual objects in each class. For example, if someone tells you that *Hamlet* is a tragedy, you know that the title character is unlikely to survive; if you are told that a lemur is a primate, then you can conclude it has an internal skeleton, is warm-blooded, and is omnivorous, even if you've never heard of lemurs before. The method of classification is an essential tool for organizing knowledge in every field.

A classification scheme can be viewed as a collection of boxes into which you are going to sort your objects. There are two requirements: every object needs to belong in some box, and no object can belong in more than one box. So you would not want to try to classify animals into "warm-blooded" and "have internal skeletons" because some animals (like spiders) would belong in neither box, and others (like humans) would belong in both. We need to make sure that our classification by symmetries satisfies these two requirements. So reaching our Final Goal means answering two questions:

Question 1. *What are the possible symmetries for two dimensional patterns?*

Question 2. *How do we tell whether two patterns have the same symmetries?*

We will begin by studying finite designs, such as logos and religious symbols, and then move on to patterns which can be extended indefinitely in one or more directions.

1.2 Necessary Tools and Materials

Journal/Notebook, Ruler, Compass, Protractor, Tracing Paper, Scissors

2 Symmetries of Finite Figures

A *finite figure* is a two-dimensional shape which is *bounded* - in other words, it can be contained in some (possibly very large) circle. We are surrounded by finite figures: alphabetical characters, company logos, religious symbols and mousepad designs are all examples of finite figures. To get a feel for what kinds of symmetry a finite figure can have, we're going to look at examples that we see (and ignore) every day - hubcaps. First, let's remember how we're defining a *symmetry* of a figure or pattern.

Definition 3. *A* symmetry *of a pattern is a rigid motion of the plane which maps the pattern to itself.*

Recall that a *rigid motion* is (informally) something you do that doesn't stretch or compress the plane, so everything stays the same size.

Task 2. *What are some rigid motions in two dimensions?*

Task 3. *You have been provided with a set of pictures of hubcaps.*

1. *Group the hubcaps according to their symmetries. What are the defining characteristic of each group?*
2. *Find another group and share your classifications. Can you find a classification that incorporates both your ideas?*
3. *Share your classification with the class.*

Task 4. *Look at each group of hubcaps in the classification from Task 3. What motions can you do to each hubcap in the group that leave it looking the same (ignoring details like logos and lugnut holes that are not really part of the design)? Do the hubcaps in the same group have the same motions? Do the hubcaps in different groups have different motions? These motions are the* symmetries *of the figures. List* all *the symmetries for each group of hubcaps.*

Definition 4. *Two figures or patterns have the same* **symmetry type** *if they have the same collection of symmetries. In other words, any motion which is a symmetry for one figure is also a symmetry for the other, and vice versa.*

So each group of hubcaps in the classification from Task 3 is a group of finite figures with the same symmetry type. But it's awkward to have to refer to a symmetry type by listing all the symmetries of those objects – it's much more convenient to have a *name* which stands in for that list (the way "mammal" stands in for "warm-blooded, bears live young, produces milk, endoskeleton, etc."). A good name should be relatively short, easy to remember, and help remind you about the characteristics of that group.

Task 5. *With a partner, find a name for each group in the classification we found in Task 3. Share your names with the class, and agree on a common naming system.*

In the next task, you are asked to find more examples of figures with the same or different symmetry types.

Task 6. *Draw six different figures in three pairs, so that each pair has the same symmetry type, but the three pairs have different symmetry types. List all the symmetries for each pair. Can you classify these figures using the scheme we developed for the hubcaps?*

Task 7. *A* regular n-gon *is a polygon with n sides so that all the sides are the same length, and all the interior angles are the same. So a regular 3-gon is an equilateral triangle, a regular 4-gon is a square, and so on. Figure 1 shows regular n-gons for $n = 3, 4, 5, 6$.*

1. *List all the symmetries for the regular n-gons in Figure 1, as you did for hubcaps in Task 4.*
2. *How many symmetries does a regular n-gon have?*

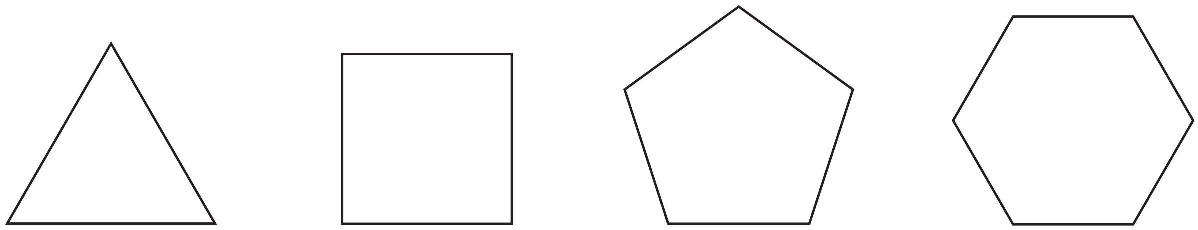


Figure 1: Regular n-gons for $n = 3, 4, 5, 6$.

3. *Describe the symmetries of a regular n-gon.*

4. *What is the symmetry type of a regular n-gon? (Use the naming system developed in class in Task 5.)*

Mathematics is about making *conjectures* (i.e. educated guesses about the answer to a question), and then working to determine whether or not the conjecture is true. We have now done enough examples to make some reasonable conjectures.

Task 8. *Conjecture answers to the following questions: What kinds of rigid motions can be symmetries of finite figures? What are the possible symmetry types for finite figures?*

Task 9. *Take pictures of 10 finite figures (other than hubcaps) you find at home or on campus, which you think exhibit symmetry. What is the symmetry type of each figure? Which of your pictures have the same symmetry type? Your pictures should include at least three different symmetry types, and at least one pair of pictures with the same symmetry type.*

2.1 Reflections and Rotations

We found that all of the symmetry types of our finite figures so far could be classified using reflections and rotations. Are these all we need? To determine this, we first need to have a clear understanding of what reflections and rotations are, and how to define them.

2.1.1 Reflections

We will begin with reflections. Reflection symmetry is also often called *mirror symmetry*, or *bilateral symmetry*. In the next Task you will construct a design with mirror symmetry.

Task 10. *1. Fold a sheet of tracing paper in half. Open the paper and label the fold line m. This line m will be your line of reflection (or* mirror line*).*

2. Draw a simple doodle on one half of the paper that touches, but doesn't cross, the line m. Label three points on your doodle (where it does not touch m) W, X, and Y.

3. Refold your tracing paper along line m, and trace the image of your doodle on the other side. Locate the images of points W, X, and Y and label these points W', X', and Y'.

4. Open your paper and draw line segments $\overline{WW'}$, $\overline{XX'}$ and $\overline{YY'}$.

5. With your partner, answer the following questions.

(a) What do you notice about the segments $\overline{WW'}$, $\overline{XX'}$ and $\overline{YY'}$ relative to m?

(b) Use a ruler to measure the (shortest) distances from W, X and Y to m.

(c) Use a ruler to measure the (shortest) distances from W', X' and Y' to m.

(d) What do you notice about the answers to (b) and (c)?

We're now ready to state the formal definition of a reflection.

Definition 5. *A* **reflection across line** *m is a motion which fixes all points on m, and sends any other point Q to a point Q' such that m is the perpendicular bisector of the line segment $\overline{QQ'}$. m is called the* **mirror line**. *An example is shown in Figure 2.*

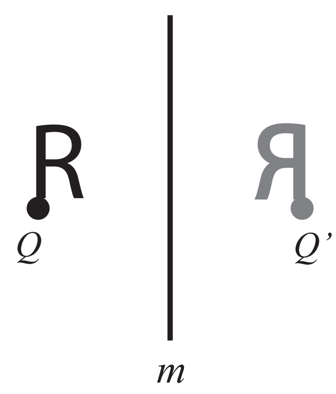


Figure 2: An example of a reflection in the line m. The original figure is in black, and the image is in gray.

Task 11. *Restate the definition of a reflection using your own words and possibly a picture. In particular, what is meant by saying m is the "perpendicular bisector" of $\overline{QQ'}$?*

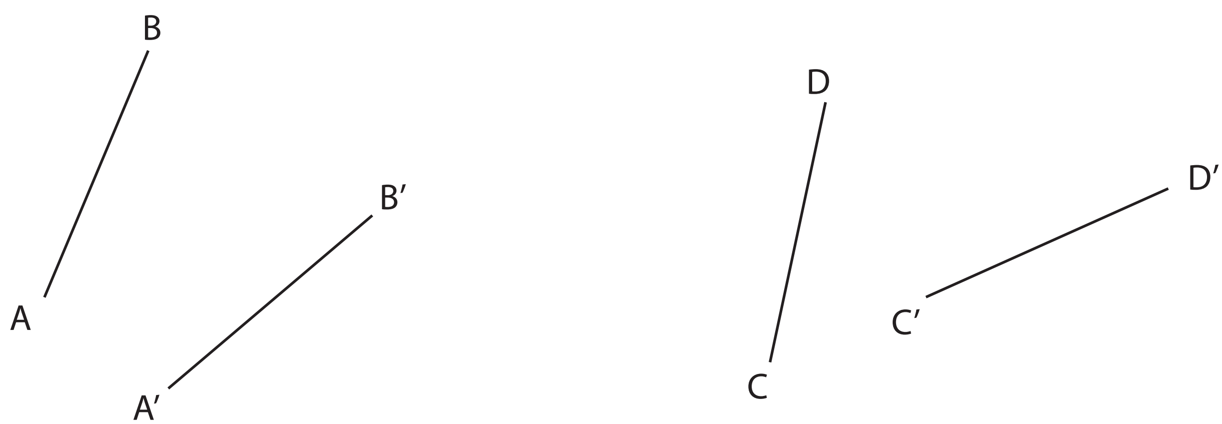
Task 12. *Do the Worksheet on Reflections.*

Task 13. *How can you tell if two line segments are reflections of each other? Give a list of necessary criteria.*

Worksheet on Reflections

1. Using a ruler and something with a right angle (such as the end of your ruler or a sheet of paper), find the line of reflection that takes P to P'.

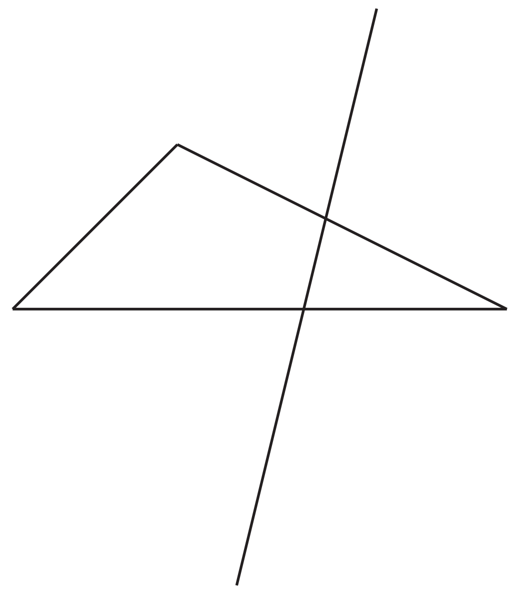
2. Consider the two pairs of line segments shown below.



   (a) Is there a single line of reflection that takes $\overline{AB}$ to $\overline{A'B'}$? If so, find it; if not, explain why not.

   (b) Is there a single line of reflection that takes $\overline{CD}$ to $\overline{C'D'}$? If so, find it; if not, explain why not.

3. Reflect the triangle over the line using a ruler and right angle.



2.1.2 Rotations

Now we will turn to rotations. In the next Task you will construct designs which are symmetric under rotation by $\frac{1}{2}$-turn (180°), $\frac{1}{4}$-turn (90°) and $\frac{1}{3}$-turn (120°).

Task 14. *Begin this task by drawing an empty square in your notebook (make it large enough that you can draw a design inside that you can easily trace). Draw a horizontal line across the middle of the square. In the middle of the line, mark a point and label it O. Now, on a sheet of tracing paper, trace the rectangle, line and point, and again label the point O.*

1. *Draw a simple doodle inside the square, to one side of the point O. Mark a point A on the doodle. Place the tracing paper on top of the doodle, matching up the horizontal lines and points O. Trace your doodle and label point A corresponding to the original A. Rotate your tracing paper 180°, keeping O fixed. Trace your doodle again and label point A' corresponding to the original A. You now have a figure with 180° rotational symmetry.*

2. *On your tracing paper, measure the distances from A and A' to O. What do you notice?*

3. *In a blank space on your tracing paper (or on a new sheet of tracing paper), trace another copy of the rectangle, line, point O, and doodle from your notebook. Now rotate the tracing paper 90°, keeping O fixed. You can use a right angle (such as the end of your ruler) or a protractor to measure the angle between the line on the tracing paper and the line in your notebook, to make sure it is 90°. Now trace the doodle again. Repeat this twice more, so that you end up with four copies of your doodle arranged around O. On each copy of the doodle, mark the point corresponding to A. Measure the distances from these points to O. What do you notice?*

4. *In a blank space on your tracing paper (or on a new sheet of tracing paper), trace another copy of the rectangle, line, point O, and doodle from your notebook. Now rotate the tracing paper 120°, keeping O fixed. You will need to use a protractor to measure the angle between the line on the tracing paper and the line in your notebook, to make sure it is 120°. Now trace the doodle again. Repeat this once more, so that you end up with three copies of your doodle arranged around O. On each copy of the doodle, mark the point corresponding to A. Measure the distances from these points to O. What do you notice?*

Now we can give our formal definition of a rotation.

Definition 6. *A* **rotation about point** O **by** $N°$ *is a motion which fixes point O, and sends any other point P to a point P' such that the measure of the angle $\angle POP'$ is $N°$ (measured counter-clockwise), and the distance between P' and O is the same as the distance between P and O. If there is a fraction $\frac{p}{q}$ so that $N = \frac{p}{q}(360)$, then a rotation by $N°$ is also referred to as a rotation by $\frac{p}{q}$ full turns. O is called the* **center of rotation***, or the* **rotocenter***. N is called the* **angle of rotation***. An example is shown in Figure 3.*

Task 15. *Restate this definition using your own words and possibly a picture.*

In Task 14 you used a known center of rotation and angle of rotation to rotate points. Now we want to work backwards and, given a point and its image under a rotation, find a possible center of rotation and angle of rotation. From Definition 6, we know that the center O of a rotation taking a point P to another point P' must be the same distance from both P and P'. How do we find such a point? One way is to use circles. A circle of radius, say, one inch centered at P consists of all the points which are one inch from P; similarly, a circle of radius one inch centered at P' consists of all the points which are one inch from P'. A point where these circles cross will be one inch from *both* P and P', as shown in Figure 4, and so can be a center for a rotation taking P to P'. In fact, we can see from Figure 4 that these circles will cross in *two* points, O_1 and O_2. Both of these are rotocenters for rotations taking P to P' (in fact, rotations by the same angle, but in opposite directions).

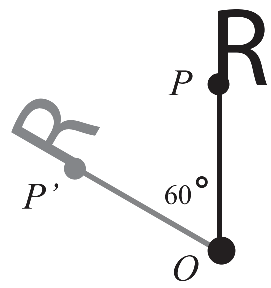


Figure 3: A rotation by 60° counter-clockwise around the point O. The original figure is in black, and the image is in gray.

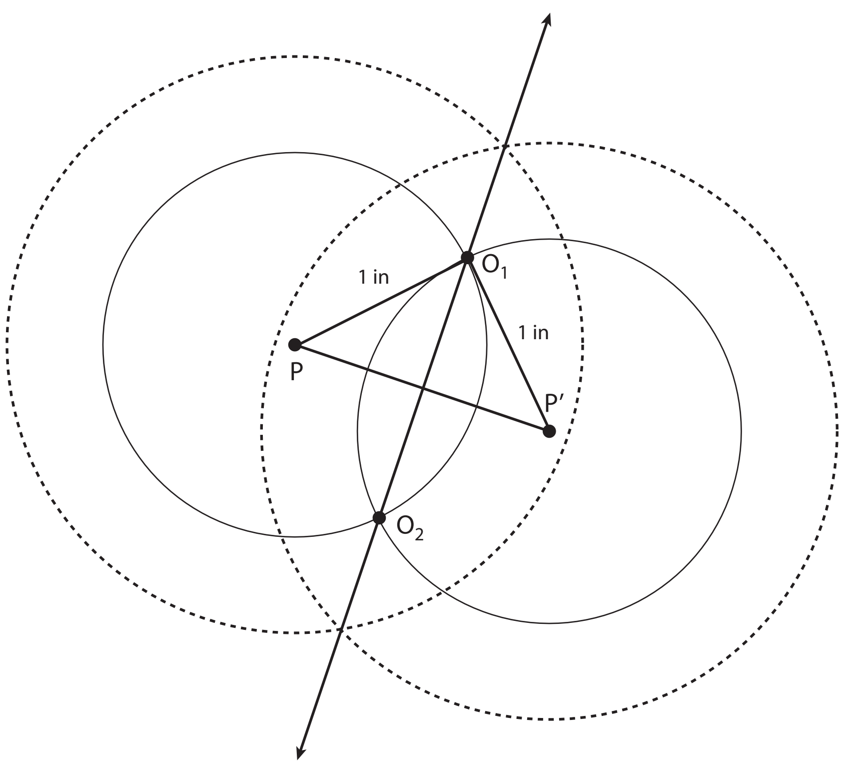


Figure 4: Finding the center for a rotation taking P to P'.

But there was nothing special about a radius of one inch. We could also draw circles of radius 1.5 inches – these are the dotted circles in Figure 4. These circles will also cross in two points. Remarkably, these two points are also on the line through O_1 and O_2! In fact, no matter what radius we choose, the two circles will intersect in two points on this line (if they intersect at all).

But what line is this? If we look again at Figure 4, we see this line is exactly the perpendicular bisector of the line segment $\overline{PP'}$! We have rediscovered the following geometric fact:

Theorem 1. *Given two points P and P', the set of points equidistant from P and P' is the perpendicular bisector of $\overline{PP'}$.*

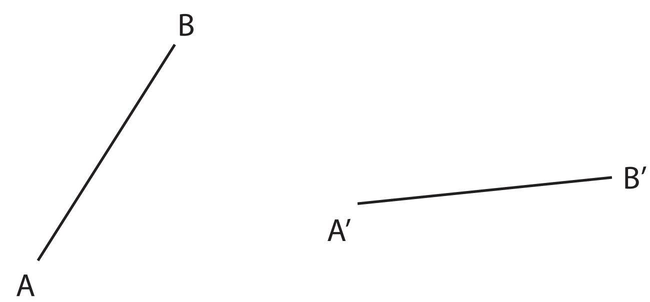
Extra Credit Task 1. *Using high-school geometry, prove Theorem 1. Hint: Use the SSS congruence theorem for triangles.*

This means that any point that is the rotocenter for some rotation taking P to P' is on the perpendicular bisector of $\overline{PP'}$; conversely, any point on the perpendicular bisector is the rotocenter of a rotation taking P to P'.

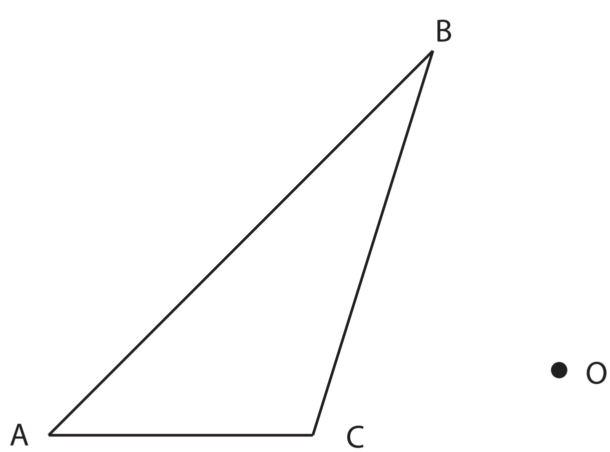
Task 16. *Do the Worksheet on Rotations.*

Worksheet on Rotations

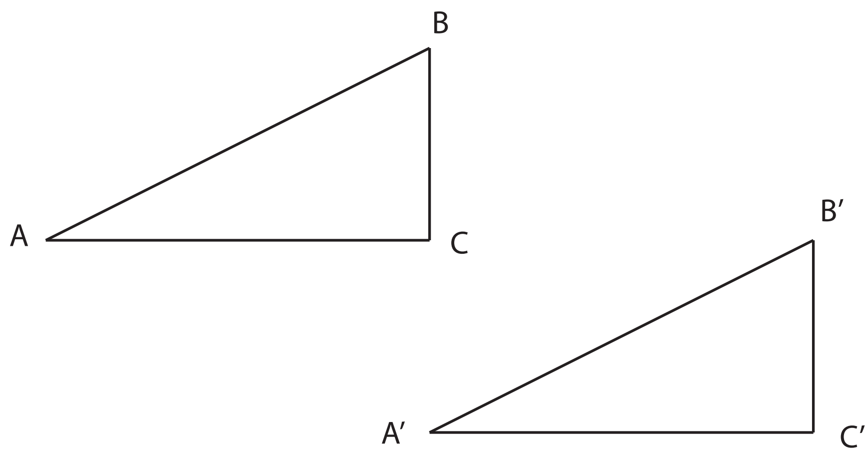
1. Find the center of rotation and angle of rotation for the rotation that takes the line segment $\overline{AB}$ to $\overline{A'B'}$ below. [Hint: The rotation will have to *both* take A to A' and B to B' - what point works as a center of rotation for both?]



2. Rotate the triangle below 100° clockwise about O.



3. Is there a rotation which takes $\triangle ABC$ to $\triangle A'B'C'$? If so, find the center and angle of rotation; if not, explain why not.



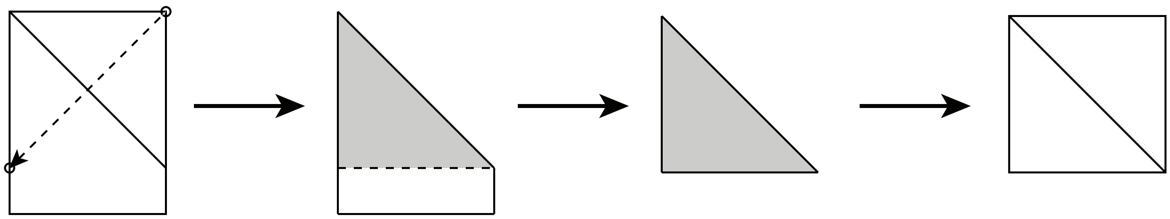


Figure 5: Instructions for folding a square.

2.2 Regular Polygons and Paper Snowflakes

Now let's apply what we've learned about rotations and reflections to some more examples of finite figures. In Task 7 you looked at pictures of some regular polygons. But, to really understand their symmetries, it's better to have a physical version you can actually move around. Recall that a regular polygon is a figure made with straight sides, such that all the sides are the same length, and all the interior angles are the same. Some of the simplest regular polygons can be constructed without any complicated measurements or tools, just by folding paper. To make them a bit more interesting (and decorative) we will use our polygons to make paper snowflakes. Real snowflakes usually have six points, due to the particular crystalline structure of frozen water; we are not so restricted, so we will make paper snowflakes with 3, 4 and 8 points, as well as the standard 6.

2.2.1 Squares

We will start with standard sheets of $8\frac{1}{2}$ inch by 11 inch paper. The simplest regular polygon to fold is the square. All you need to do is fold the upper right corner of the paper down to the left side, along a crease going through the upper left corner, as shown in Figure 5. Cut off the extra strip along the bottom, and you're left with a square.

Task 17. *Explain why this construction produces a square. In particular, how do we know (without using a ruler!) that all four sides are the same length?*

Task 18. *Make a square. Label the corners on one side A, B, C, D (in order clockwise), and on the other side A, B, C, D in order* counter-*clockwise (so that A is on the other side of the paper from A, B from B, and so forth).*

1. *What are the rotation symmetries of the square? Denote each rotation by its angle of rotation* counter-clockwise. *For each rotation symmetry, record what happens to the labels A, B, C, D by drawing a picture of the square after the motion. The "do-nothing" motion, in which nothing moves, is considered a rotation by $0°$ (and has the same effect as rotation by $360°$).*

2. *If you fold the square along a mirror line for a reflection symmetry, the square will exactly fold onto itself – one such line is the crease you made in creating the square. Find the other mirror lines and fold the square to make creases along each line. You should have 4 crease lines. Use a pen or pencil to trace the lines (on both sides of the square), and mark the point where they cross. Label the point where they cross O on the front and back.*

3. *For each reflection symmetry, record what happens to the labels A, B, C, D by drawing a picture of the square after the motion. We will name the reflection symmetries as follows (assuming the square begins with its sides aligned vertically and horizontally):*

   - m_1 = *reflection in the* **vertical line through** O

- m_2 = *reflection in the* **diagonal line from the lower left corner to the upper right corner**
- m_3 = *reflection in the* **horizontal line through** O
- m_4 = *reflection in the* **diagonal line from the upper left corner to the lower right corner**

4. *What happens to O when you do a rotation or reflection symmetry of the square?*

Now we will explore what happens when we combine symmetries of the square. If we do a sequence of rotations and reflections, is the final position the square something new, or could we have achieved the same effect with just a single rotation or reflection?

Task 19. *Perform each of the following sequences of motions on the square. In each case, is it possible to match the final result (the arrangement of the labels A, B, C, D) with the result of one of the rotations or reflections from Task 18? If so, say which one.*

1. *Rotate* $90°$ *counterclockwise, perform reflection* m_2*, and then rotate* $180°$ *clockwise.*
2. *Perform reflection* m_3*, and then reflection* m_2*.*
3. *Perform reflection* m_2*, and then reflection* m_3*. (Was the result the same as in the previous part?)*
4. *Perform reflection* m_1*, rotate* $270°$ *clockwise, then perform reflection* m_4*.*
5. *Rotate* $90°$ *counterclockwise, then rotate another* $90°$ *counterclockwise, perform reflection* m_2*, rotate* $180°$ *counterclockwise, and perform reflection* m_4*.*

Do you think it's possible to get a new symmetry (i.e. one we didn't find in Task 18) by combining the rotation and reflection symmetries? How many symmetries does the square have?

Task 20. *Do a few more examples like Task 19 on your own, and answer the following questions.*

1. *When you combine two rotations, is the resulting symmetry a rotation or a reflection? If you do them in the other order, do you get the exact same symmetry?*
2. *When you combine two reflections, is the resulting symmetry a rotation or a reflection? If you do them in the other order, do you get the exact same symmetry?*
3. *When you combine a rotation and a reflection, is the resulting symmetry a rotation or a reflection? If you do them in the other order, do you get the exact same symmetry?*

Extra Credit Task 2.

1. *Show that every symmetry of the square can be written as a combination of the* $90°$ *clockwise rotation and the reflection* m_1 *in the vertical line (possibly using each one several times). For brevity, we will call these two motions* r *(for rotation) and* m *(for mirror), respectively.*
2. *The set* $\{r, m\}$ *is called a* generating set *for the symmetries of the square, and* r *and* m *are called* generators. *But this is not the only possible generating set. For a set* G *of motions to be a generating set for the symmetries of a square, we need three things: (1) every motion in* G *is a symmetry of the square, (2) every symmetry of the square is a combination of the motions in* G *and (3) none of the motions in* G *are a combination of the other motions in* G*. Find other generating sets for the symmetries of a square. Can you find all the possible generating sets?*

Now let's use our square to make a four-pointed paper snowflake.

Task 21. *Make a new square (in fact, you will probably want several), as in Figure 5. Fold it along all its mirror lines as in Figure 6 to get a right triangle. Now experiment with cutting pieces off of the triangle and looking at the result when the snowflake is unfolded.*

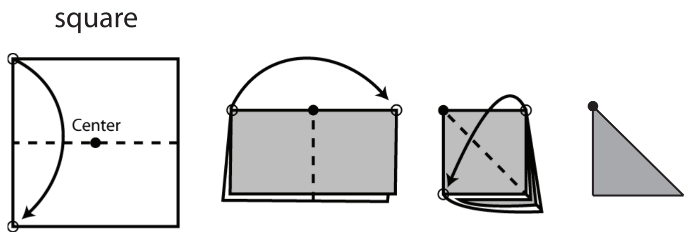


Figure 6: Instructions for making a four-pointed snowflake.

1. *What happens when you cut a piece from the hypotenuse of the triangle? The leg adjacent to the center of the square? The remaining side?*
2. *What happens when you cut off the corner containing the center of the square? The corner with the right angle? The third corner?*
3. *Sketch a couple of the resulting snowflakes showing what happens.*

Task 22. *What are the symmetry types of the four-pointed snowflakes you have created? Could you create "snowflakes" with a different symmetry type by folding the square along only some, but not all, of the mirror lines before cutting it? If so, what different symmetry types are possible?*

The figure you have after you've made your cuts, but before you open up the snowflake, is called the *fundamental domain* for the final snowflake. A fundamental domain of a design is an elementary design that can be copied, rotated and flipped to make the entire design. A four-pointed snowflake has eight copies of its fundamental domain; each region between two adjacent mirror lines is a copy of the fundamental domain. You can imagine placing the folded snowflake on any of the eight regions of the open snowflake.

Task 23. *Color one fundamental domain on one of your four-pointed snowflakes, and sketch the result. What are the fundamental domains for the "snowflakes" with other symmetry types you created in Task 22?*

2.2.2 Equilateral Triangles

We can also construct an equilateral triangle using paper folding, starting with a square, as shown in Figure 7. In the diagrams, dashed lines indicate the creases for the folds. The bottom side of the square will be the bottom side of the equilateral triangle. First, fold the square in half and open it up again. Fold one corner so that it lands on the center crease and the fold goes through the adjacent corner. Then fold again along the line between the point on the center crease and the adjacent corner. Open all of the folds so that the square lies flat. Fold and unfold along the line connecting the corner and the point on the center line so that the result gives an equilateral triangle. Now you can cut along the creases to cut out an equilateral triangle.

Task 24. *Make an equilateral triangle using the instructions above. Without using a ruler, explain how you know that the figure you've constructed is, in fact, an equilateral triangle. To do this, you need to explain how you know all three sides are the same length.*

Task 25. *Label the corners of the triangle A, B, C (clockwise) on one side, and A, B, C (counterclockwise) on the other side (so that A is on the other side of the paper from A, and so forth).*

1. *As you did for the square in Task 18, part 2, find the mirror lines and the point where they cross. In this case, you should have 3 mirror lines.*
2. *Record the effect on the corners of each (clockwise) rotation and reflection symmetry of the triangle by drawing a picture, as you did for the square in Task 18.*

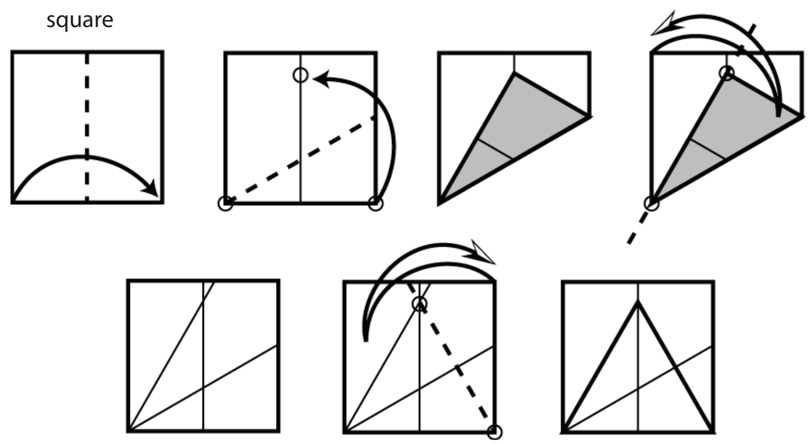


Figure 7: Instructions for making an equilateral triangle.

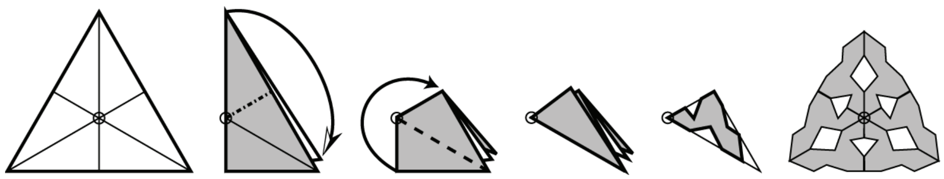


Figure 8: Instructions for making a three-pointed paper snowflake.

Task 26. *Do the answers you found to the questions in Task 20 hold for the triangle as well? If not, how are the answers different?*

Now let's use our equilateral triangles to make three-pointed paper snowflakes.

Task 27. *Make a new triangle (or several), as in Figure 7. Fold it along all its mirror lines as in Figure 8. As in Task 21, experiment with the results of cutting the folded triangle in different places, and sketch a couple of the resulting snowflakes to illustrate what you find.*

Task 28. *What are the symmetry types of the three-pointed snowflakes you have created? Could you create "snowflakes" with a different symmetry type by folding the triangle along only some, but not all, of the mirror lines before cutting it? If so, what different symmetry types are possible?*

As with the four-pointed snowflakes, the figure you have after you've made your cuts, but before you open up the snowflake, is called the *fundamental domain* for the final snowflake. A three-pointed snowflake has six copies of its fundamental domain.

Task 29. *Color one fundamental domain on one of your four-pointed snowflakes, and sketch the result. What are the fundamental domains for the "snowflakes" with other symmetry types you created in Task 28?*

2.2.3 Regular Hexagons and Octagons

It is also fairly easy to construct a regular hexagon and a regular octagon by paper folding (see Figures 9 and 10). Notice that in each case we begin with a SQUARE. As before, the dashed lines indicate the creases for each fold.

Task 30. *Make a regular hexagon, following the instructions in Figure 9. Explain how you know (without using a ruler) that the result of the construction is, in fact, a regular hexagon.*

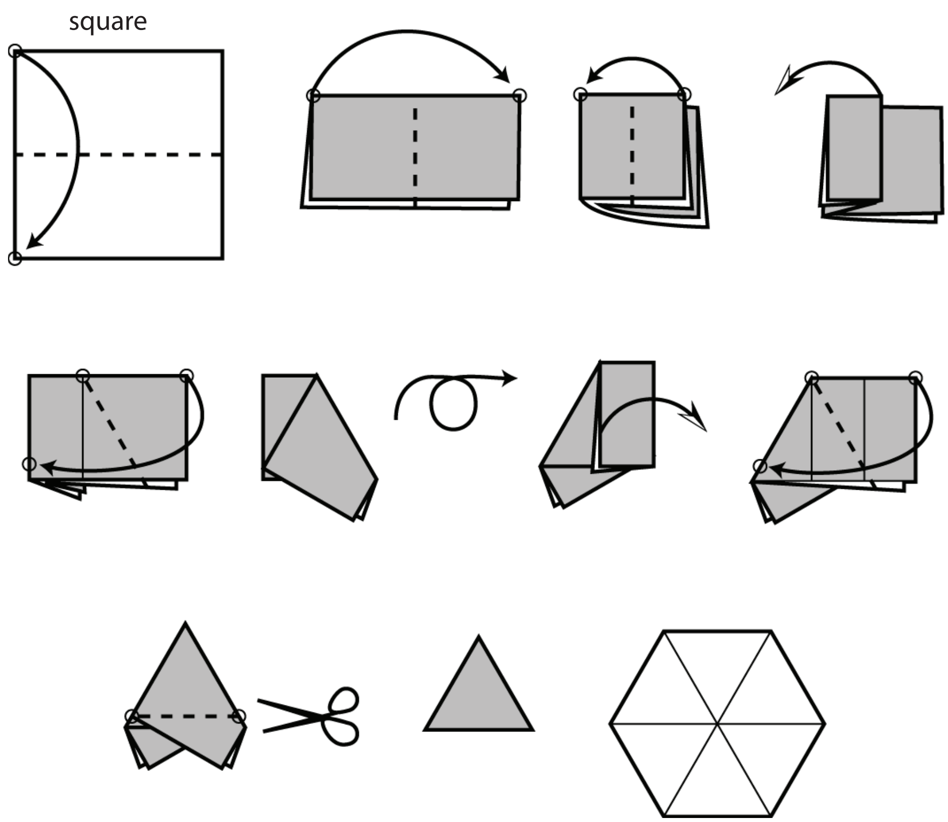


Figure 9: Instructions for folding a regular hexagon.

Task 31. *Use your regular hexagon to complete this task.*

1. *Label the corners of the hexagon, as you did for the square and triangle in Tasks 18 and 25.*
2. *What are the rotation symmetries of the hexagon? Record the effect on the corners of each (clockwise) rotation and reflection symmetry of the hexagon.*
3. *Do the answers you found to the questions in Task 20 hold for the hexagon as well? If not, how are the answers different?*

Task 32. *Use a regular hexagon (constructed as in Figure 9) to make a six-pointed paper snowflake (a classic paper snowflake). Color one fundamental domain on your snowflake.*

Task 33. *What is the symmetry type of the six-pointed snowflakes you have created? Could you create "snowflakes" with a different symmetry type by folding the hexagon along only some, but not all, of the mirror lines before cutting it? If so, what different symmetry types are possible?*

Extra Credit Task 3. *What is the symmetry type of an n-pointed snowflake formed from a regular n-gon by folding along all mirror lines? What are the possible symmetry types of the "snowflakes" that can be created by by folding along some, but not all, of the mirror lines?*

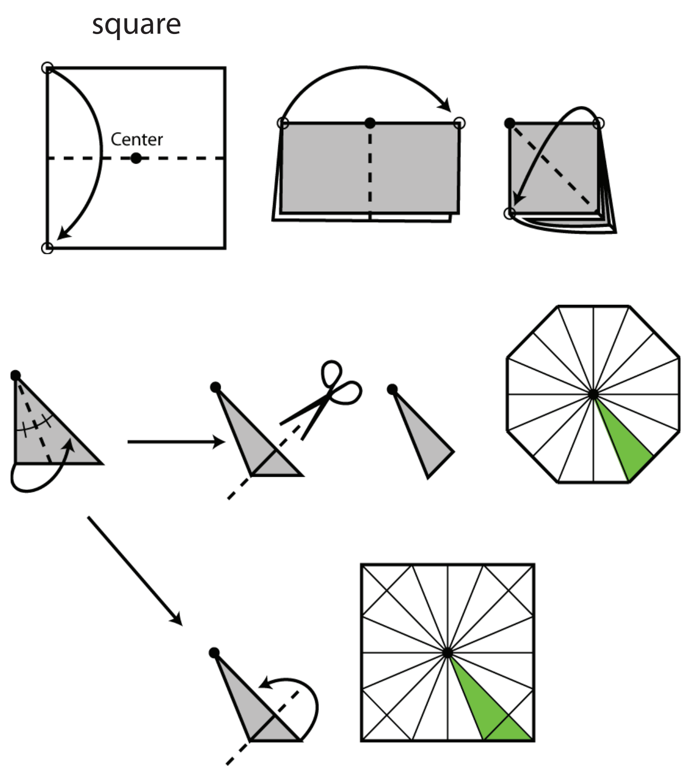


Figure 10: Instructions for folding a regular octagon.

Extra Credit Task 4. *Make a regular octagon, following the instructions in Figure 10. Explain how you know (without using a ruler) that the result of the construction is, in fact, a regular octagon.*

Extra Credit Task 5. *Use your regular octagon to complete this task.*

1. *Label the corners of the octagon, as you did for the square and triangle in Tasks 18 and 25.*
2. *What are the rotation symmetries of the octagon? Record the effect on the corners of each (clockwise) rotation and reflection symmetry of the octagon.*
3. *Do the answers you found to the questions in Task 20 hold for the octagon as well? If not, how are the answers different?*

Extra Credit Task 6. Art project: Snowflakes. *Experiment with folding and cutting snowflakes in different ways. Make a collage of at least 3 different snowflakes. Type a five paragraph essay explaining what you notice. Be sure to discuss the way you folded your paper and the type of repeats in the snowflakes.*

2.3 Classification of Rigid Motions with Fixed Points

2.3.1 Rigid Motions

Rotations and reflections are both examples of *rigid motions*. Now we're ready to consider a more formal definition of *rigid motion* than we had before.

Definition 7. *A* **rigid motion** *T sends any point P in the plane to another point $T(P)$, such that for any points P and Q with images $T(P)$ and $T(Q)$, the distance between $T(P)$ and $T(Q)$ is the same as the distance between P and Q, as shown in Figure 11.*

An Explanation of the Notation: There's some new notation here, which we need to discuss. A rigid motion is a *motion*, which means it *moves* things. The things it moves are the points of the two-dimensional plane (an infinite flat surface). So for a given rigid motion T and a given point P, performing the motion T moves P to some other point on the plane (or possibly leaves it where it is). So we often want to talk about "the point (i.e., the location in the plane) to which P is moved by the motion T." Rather than this awkward phrase, we call this point $T(P)$ (pronounced "T of P"). We may also refer to this as "the image of P under T."

Figure 11 shows an example of a rigid motion T, illustrated by its effect on points P, Q, R and S. So here $T(P)$ is the place where P is moved by T, $T(Q)$ is the place where Q is moved by T, and so forth. The distances between the pairs of points are indicated by a, b, c, d and e; in each case, the distance between the images of the points under T is the same as the distance between the original points. Also, the fact that all the distances are the same also forces all the angles to be the same. So the measure of angle $\angle QPS$ is the same as the measure of angle $\angle T(Q)T(P)T(S)$.

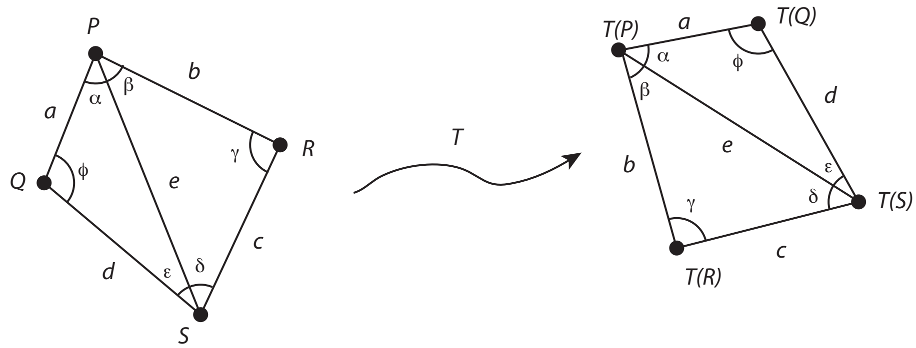


Figure 11: An example of a rigid motion. Notice that the distance between $T(P)$ and $T(Q)$ is the same as the distance between P and Q.

> **Extra Credit Task 7.** *Using high school geometry, prove that rotations and reflections are rigid motions. [Hint: Side-Angle-Side Congruence Theorem]*

If we have a rigid motion, how do we determine *what* rigid motion it is? Part of the problem is knowing how to describe a given rigid motion – certainly, it's not practical to describe what happens to every point in the plane! But perhaps we don't need this much. The following theorem shows that once you know what a rigid motion does to three points, you can figure out what it does to every other point. So you can unambiguously describe any rigid motion by describing what happens to three points!

Theorem 2. *Any rigid motion is determined by its action on three non-collinear points (i.e., three points that are not all on the same line).*

Intuitive Proof: If we know what a motion T does to points A, B and C, then for any fourth point P, the distances from $T(P)$ to $T(A)$, $T(B)$ and $T(C)$ must be the same as the distances from P to A, B and C. But if $T(A)$, $T(B)$, $T(C)$ are not on the same line, there will be only point which is the correct distance from all three points. This point must be $T(P)$, so the image of every other point is determined. ☐

Proof. Let's say we have a rigid motion T and three points A, B, C with images $T(A), T(B), T(C)$. Given any point P, we need to show that we can find $T(P)$.

Since we know where P is to start with, we know how far P is from A, B, C. Let a be the distance from P to A, b be the distance from P to B and c be the distance from P to C, as shown in Figure 12.

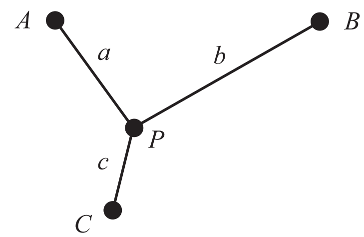


Figure 12: a, b and c are the distance from P to A, B and C, respectively.

Since T is a rigid motion, a is also the distance from $T(P)$ to $T(A)$. This means that $T(P)$ is somewhere on the circle of radius a centered at $T(A)$ (see Figure 13). Moreover, b is also the distance from $T(P)$ to $T(B)$, and so $T(P)$ is also on the circle of radius b around $T(B)$ (see Figure 13). In general, these two circles meet in just two points, so $T(P)$ must be one of these two points. In fact, these two points are mirror images of each other across the line through $T(A)$ and $T(B)$. Since $T(C)$ is not on this line (because A, B, C are not collinear), the two points are different distances from $T(C)$. The one which is c units from $T(C)$ will be $T(P)$, as shown in Figure 13. ☐

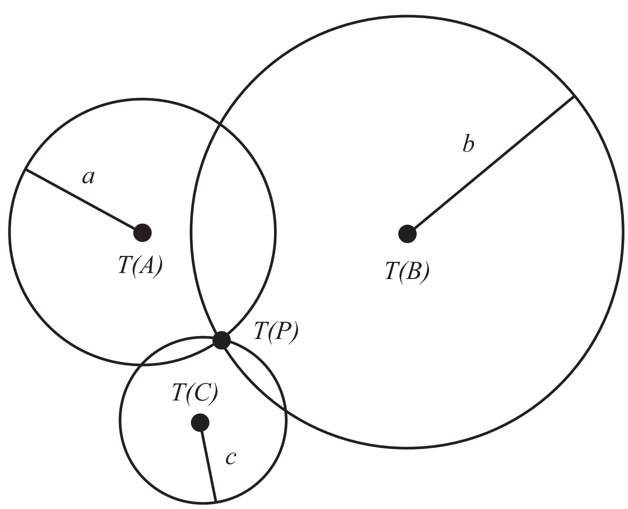


Figure 13: $T(P)$ is the point where the three circles intersect.

A convenient way to draw three non-collinear points is by using a capital "R" (the three points are the upper left corner, and the ends of the two legs). If we reflect an "R", we get the cyrillic letter "Я", pronounced "ya". So any rotation takes an R to an R, while any reflection takes an R to a Я.

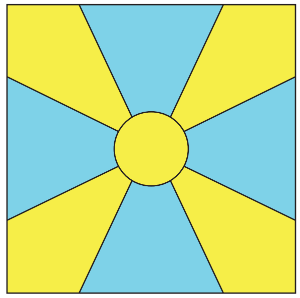


Figure 14: A finite figure which can extend to infinity.

We know that rotations and reflections are rigid motions which can be symmetries of finite figures. Are there any other rigid motions which can be a symmetry of a finite figure? To answer this, we first need to revisit our definition of a finite figure. We originally defined a finite figure as a shape which is *bounded*. This meant that if the figure were moved in any direction, we would be able to see the difference, so that kind of motion would not be a symmetry. But there are unbounded designs which also have this property. For example, the design in Figure 14 can easily be extended out to infinity – but we could still tell if it had been moved because there is a well-defined center to the design. So whether we limit the design to a small space, or extend it to infinity, it has the same symmetries. This suggests a new, slightly broader, definition of a finite figure – one that is based on symmetries, rather than shape or size. Essentially, a finite figure is a design which has a well-defined *center*. To make this more precise, we give the following new definition for a finite figure.

Definition 8. *A* **finite figure** *is a figure that contains a point C which is fixed by every symmetry of the design. C is called the* **center** *of the finite figure.*

In this next two sections we will show that rotations and reflections are the *only* rigid motions that fix a point, and hence the only rigid motions that can be symmetries of a finite figure. To do this, we need to prove two things:

1. Any combination of rotations and reflections (that all fix a given point) has the same effect as a single rotation or reflection.
2. Any rigid motion that fixes a point is a combination of rotations and reflections that all fix that point.

2.3.2 Combining Rotations and Reflections

If we perform one rigid motion and follow it with another rigid motion, the result will still be a rigid motion. In Task 20 you saw that every combination of the rotation and reflection symmetries of a square resulted in another rotation or reflection. In this section, we will show that this is true in general, as long as there is a point which is fixed by all the motions being combined.

Notation: If we have two rigid motions S and T, and a point P, then $T(P)$ is the image of P under motion T. Similarly, $S(T(P))$ is the image of $T(P)$ under motion S. Hence $S(T(P))$ is the image P after we *first* perform motion T and *then* perform motion S. We will often write this more compactly as $ST(P)$, and use ST to refer to the *combined* motion which results by performing *first* T and *then* S. So it's important to remember that when we write combinations of motion in this way, we are reading the order of the motions from right to left, *not* from left to right.

Task 34. *In your workbook, draw an "R" and pick some point P on the page (not on the "R"). Use a ruler and a protractor to rotate the "R" 40° counter-clockwise about P (by rotating the three corners of the* R*,* *and then connecting the images); then rotate the image another* $60°$ *counter-clockwise about* P*. What is the combined effect of the two rotations on the original "R"?*

Task 35. Make a conjecture: *What is the rigid motion that results by combining a rotation by an angle* α *counter-clockwise about a point* P *with a rotation by an angle* β *about* P*? How does the result depend on the order in which we do the two rotations?*

Task 36. *In your workbook, draw an "R" and pick point* P *on the page (not on the "R"). Draw a line* m *through* P *such that* m *does not pass through the "R". Use a ruler and a protractor to rotate the "R"* $60°$ *counter-clockwise about* P*; then reflect the image through* m*. Notice that the result is a reflection of the original* R*. Find the line of reflection* n *for this reflection; what is the angle between* m *and* n*? Now do the motions in the opposite order; is the mirror line of the resulting reflection the same as* n*? If not, what is the angle between this mirror line and* m*?*

Task 37. Make a conjecture: *What is the rigid motion that results by combining a rotation by an angle* α *counter-clockwise about a point* P *with a reflection across a line through* P*? How does the result depend on the order in which we do the two rigid motions?*

Task 38. *In your workbook, draw an "R" and pick point* P *on the page (not on the "R"). Draw two lines* k *and* m *through* P *(so that* m *and* k *do not pass through* R*), such that the angle between* k *and* m *is* $30°$*. Reflect the "R" in* k*, then reflect the image through* m*. Notice that the final result is a rotation of the original* R*. Where is the center of rotation, and what is the angle of rotation? Now do the reflections in the opposite order; is the rotation the same? If not, how is it different?*

Task 39. Make a conjecture: *What is the rigid motion that results by combining two reflections in lines through a point* P*, where* α *is the angle between the mirror lines? How does the result depend on the order in which we do the two reflections?*

Now let's prove our conjectures.

Theorem 3. *Say that* A *is a rotation with center of rotation* P *and angle of rotation* α*, and* B *is a rotation with center of rotation* P *(the same as* A*), and angle of rotation* β*. Then* BA *is a rotation about* P *by angle* $\beta + \alpha$*, and* AB *is the same as* BA*.*

Proof. The proof is immediate from Figure 15 (here, the rotations shown are clockwise; a similar figure can be drawn for counter-clockwise rotations). By Theorem 2, it's enough to consider what happens to points P, Q and S, and it is clear that each of these points have been rotated clockwise by an angle $\beta + \alpha$ about P. Since $\alpha + \beta = \beta + \alpha$, $AB = BA$. □

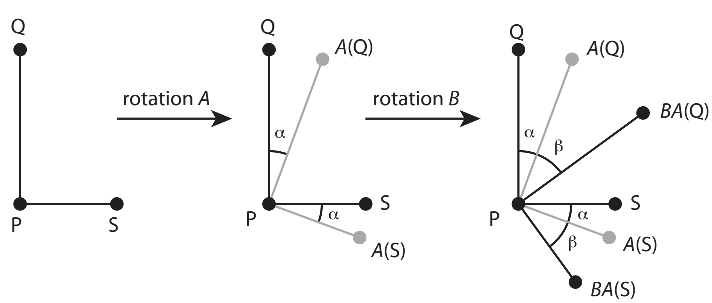


Figure 15: Combining two rotations.

Theorem 4. *Say that R is a rotation about a point P by angle α, and M is a reflection in a line m through P. Then RM is a reflection in a line n through P, where the angle between m and n is $\frac{1}{2}\alpha$. Similarly, MR is a reflection in a line n' through P, where the angle between m and n is $-\frac{1}{2}\alpha$ (i.e. n and n' are rotated from m by the same amount, but in opposite directions).*

Proof. Figure 16 shows the result of combining the reflection M in the line m with the rotation R with center of rotation P and angle of rotation α, to get the final motion RM. We can see from the figure that $RM(P)$, $RM(Q)$ and $RM(S)$ are the reflections of P, Q and S, respectively, through the line n that bisects the angle $\angle Q\,P\,RM(Q)$. So by Theorem 2, the motion RM is a reflection in mirror line n. A similar diagram shows that MR is a reflection in a line n' which is on the opposite side of m from n. □

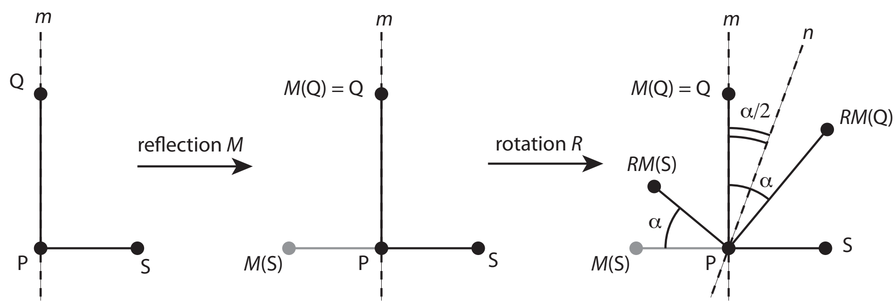


Figure 16: Combining a rotation and a reflection.

Theorem 5. *Say that M is a reflection in a line m through a point P, and K is a reflection in a line k through P, so that the angle from m to k is α. Then MK is a rotation about P by angle 2α, and KM is a rotation about P by angle -2α (i.e. by the same angle, but in the opposite direction).*

Proof. Figure 17 shows the result of combining the reflection M in the line m with the reflection K in the line k, to get the final motion KM. Since k is the bisector of angle $\angle Q\,P\,KM(Q)$, this angle has measure 2α. Similarly, the measure of angle $\angle S\,P\,KM(S)$ is the same as the measure of $\angle S\,P\,M(S)$, which is also 2α. Hence KM acts on P, Q and S as the rotation about P by angle 2α. By Theorem 2, this means KM is the rotation. A similar diagram shows that MK is the rotation by 2α in the opposite direction. □

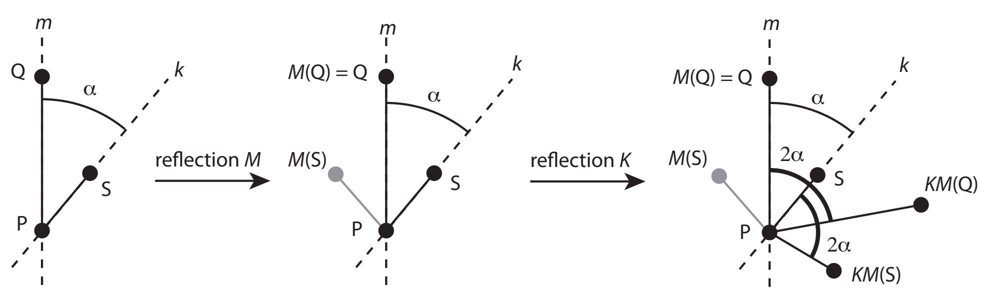


Figure 17: Combining two reflections.

Together, Theorems 3, 4 and 5 show that every combination of rotations around a point P and reflections in lines through P results in a single rotation or reflection.

2.3.3 Rigid Motions that Fix a Point

Now we turn to proving that *any* rigid motion that fixes a point P is a combination of rotations about P and reflections in lines through P.

Theorem 6. *Any rigid motion which fixes a point P is either*

1. *a rotation with center of rotation P, or*

2. *the result of combining a rotation with center of rotation P with a reflection whose mirror line goes through P.*

Intuitive Proof: Let T be the rigid motion fixing P. By Theorem 2, it's enough to consider any three non-collinear points. We pick P to be one of the points, and let Q and S be two other points. We can move Q to $T(Q)$ by a rotation about P; then either S has been moved to $T(S)$, or we can move it to $T(S)$ by reflecting in the line through P and $T(Q)$. □

Proof. Let T be the rigid motion fixing P. By Theorem 2, it's enough to consider any three non-collinear points. We pick P to be one of the points, and let Q and S be two other points, as shown in Figure 18. Let a, b and c denote the distances between each pair of points, as shown in Figure 18.

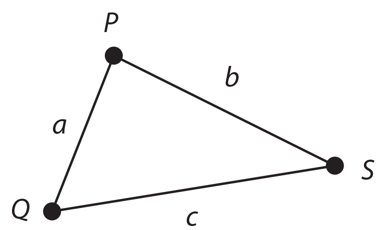


Figure 18: Three noncollinear points P, Q, S. a, b, c denote the distances between the points.

Since T is a rigid motion and $T(P) = P$, Q and $T(Q)$ are both distance a from P. So Q and $T(Q)$ are both on the circle of radius a centered at P. Let α be the measure of the angle $\angle Q\,P\,T(Q)$. Then if R is the rotation with center of rotation P and angle of rotation α, $R(Q)$ will be the same point as $T(Q)$, as shown in Figure 19.

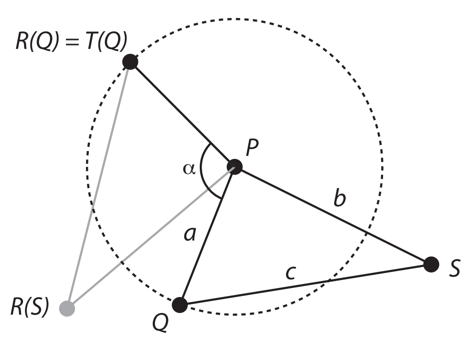


Figure 19: The result of rotating Q and S about P by angle α.

Since T is a rigid motion, the distance between $T(S)$ and P must be b (the same as the distance between S and P), and the distance between $T(S)$ and $T(Q)$ must be c (the same as the distance between S and

Q). So $T(S)$ must be on both the circle of radius b around P and the circle of radius c around $T(Q)$. Since the rotation R is a rigid motion, $R(S)$ is also on both circles. But these two circles cross in only two points. One of these points is $R(S)$, and the other is the reflection of $R(S)$ across the line through P and $T(Q)$. We will call this reflection M, so the other point of intersection is $MR(S)$, as shown in Figure 20. Note that $MR(P) = P$ and $MR(Q) = T(Q)$

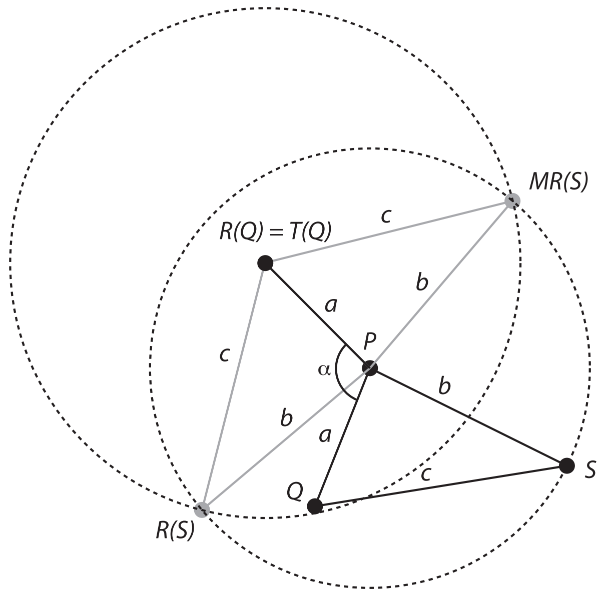


Figure 20: Reflecting $R(S)$ across the line between P and $T(Q)$.

Since $T(S)$ must be a point of intersection of the circles, either $T(S) = R(S)$ or $T(S) = MR(S)$. So, by Theorem 2, either $T = R$ or $T = MR$. □

Task 40. *Explain why every rigid motion which fixes a point is either a rotation about the fixed point, or a reflection through a line through the fixed point. [Hint: Combine Theorem 4 and Theorem 6.]*

So any motion which fixes a point is either a rotation or a reflection.

Extra Credit Task 8. *If we are willing to allow combinations of motions, rather than reducing to a single motion, then we only need to consider reflections. Prove that any rotation about a point P is the result of combining two reflections in lines through P. [Hint: this is the reverse of Theorem 5.]*

2.4 Classification of Finite Figures

Now we know that any symmetry of a finite figure has to be either a rotation or a reflection. The question becomes, which different collections of symmetries are possible?

So far, all the finite figures we've seen have had C_n or D_n as their symmetry type. Table 1 gives the symmetries in each case.

Symmetry Type	**Rotation Symmetries**	**Reflection Symmetries**
C_n	$0°, \frac{360}{n}°, 2\left(\frac{360}{n}\right)°, \ldots, (n-1)\left(\frac{360}{n}\right)°$	none
D_n	$0°, \frac{360}{n}°, 2\left(\frac{360}{n}\right)°, \ldots, (n-1)\left(\frac{360}{n}\right)°$	n reflection symmetries, where the angle between adjacent mirror lines is $\frac{180}{n}$

Table 1: Symmetries of Finite Figures with Symmetry Type C_n or D_n

Task 41. *Draw examples of finite figures with symmetry types C_3, C_5, D_4 and D_6. In each case, list the rotation symmetries of the figure. In the last two cases, draw lines on the figures to indicate the mirror lines, and find the angle between adjacent mirror lines.*

We want to prove that *any* finite figure has to have symmetry type C_n or D_n for some n. First we have to make an important observation:

Fact. *If A and B are symmetries of a finite figure F, then the combinations AB and BA are also symmetries of F. This is because A and B both move F back onto itself, so doing them one after the other will also move F back onto itself. We call this property of the symmetries of a figure* **closure***, and we say that the set of symmetries of a figure is* **closed***.*

We are first going to look at the collection of rotation symmetries for a finite figure F. We are going to show that if a figure has rotation symmetries, the angles of rotation must be all the multiples of $\frac{360}{n}°$, for some n (these are the rotation symmetries of a figure of type C_n, as in Table 1). The first step is to show that all rotation symmetries are multiples of the smallest rotation symmetry.

Theorem 7. *Say that α is the smallest possible angle of a non-trivial rotation symmetry of a finite figure F. Then the angle of any other rotation symmetry of F is a multiple of α.*

Proof. We use the method of *proof by contradiction*. In other words, we assume that our conclusion is false, and show that this would lead to an impossibility. So let's assume that *not* every rotation symmetry of F is a multiple of α. Then there is a rotation symmetry whose angle β lies between two multiples of α – i.e., for some integer k, $k\alpha < \beta < (k+1)\alpha$.

By Theorem 3, if we repeat the rotation by α a total of k times, we get a rotation by $k\alpha$. Since the set of symmetries is closed, this rotation is also a symmetry of F. Similarly, combining the rotation by angle β in one direction with a rotation by angle $k\alpha$ in the other direction gives a new rotation symmetry, with an angle of rotation of $\beta - k\alpha$. But, from our earlier inequality, we see $0 < \beta - k\alpha < \alpha$. So there is a non-trivial rotation symmetry whose angle of rotation is less than α, which contradicts the assumption that α is the angle of the smallest non-trivial rotation symmetry.

Therefore every rotation symmetry must be a multiple of α. □

Task 42. *Say that α is the angle of the smallest rotation symmetry of a finite figure F.*

1. *Explain why $360°$ must be a multiple of α (in other words, $\alpha = \frac{360}{n}$ for some n). [Hint: what happens if you rotate a finite figure $360°$?]*

2. *Use the previous part, Theorem 3 and Theorem 7 to explain why any finite figure with rotation symmetries has the same rotations as C_n, for some n (see Table 1).*

Now that we understand the possible collections of rotation symmetries, what happens if we add reflections into the mix?

Task 43. *Explain why a finite figure F with more than one reflection symmetry must have a rotation symmetry other than the "do-nothing" symmetry. [Hint: use Theorem 5 together with the fact that the symmetries of F are closed.]*

So (almost) every finite figure with reflection symmetries also has rotation symmetries. Let's look at an example of a figure with both a reflection and a rotation symmetry, and see what other symmetries it has to have.

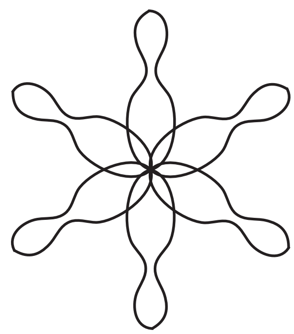


Figure 21: Figure for Task 44

Task 44. *Consider the finite figure in Figure 21.*

1. *What are the rotation symmetries of this finite figure? Let R denote the rotation symmetry with the smallest (counter-clockwise) angle of rotation.*
2. *Let M denote the reflection symmetry across the vertical line through the center of the figure. Then RM is the reflection which results from combining the reflection M with the rotation R (doing the reflection first). Find the angle between the mirror line for RM and the vertical mirror line for M. (Hint: Use Theorem 4.) Draw the mirror line for RM on the finite figure.*
3. *Find all the other mirror lines of the figure. Show that each reflection can be written as $R^k M$ for some k (here, R^k is the rotation that results from doing the rotation R a total of k times). What is the angle between each pair of adjacent mirror lines?*

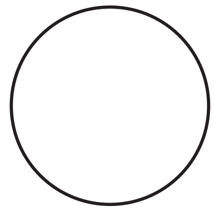
Now that we've looked at a specific example, let's consider the general case.

Task 45. *Say that a finite figure F has a reflection symmetry M, and the smallest nontrivial rotation symmetry R is by angle $\alpha = \left(\frac{360}{n}\right)^\circ$ around a point P.*

1. *Show that F also has reflection symmetries $RM, R^2M, \ldots, R^{n-1}M$, that all the mirrors cross at P, and the angle between two adjacent mirrors is $\frac{180}{n}$.*
2. *Show that F cannot have any other reflection symmetries. [Hint: show that if there is a mirror line between M and RM, then there is a rotation symmetry with an angle smaller than α.]*

Task 46. *Combine the results of Tasks 42 and 45 to explain why every finite figure has symmetry type C_n or D_n.*

There is one caveat to the result of Task 46 – namely, the circle.



Where does the circle fit into our classification scheme? The problem is that, unlike the other figures we've looked at, the circle does *not* have a smallest rotation symmetry – any rotation, however small, leaves the circle looking the same. We can see the circle as a limiting case as the angle of the smallest rotation decreases towards zero – or, since the smallest angle is $\frac{360}{n}^{\circ}$, as n increases towards infinity. So we will create another symmetry type for the circle (and other figures like it), called D_∞. (We use D_∞, rather than C_∞, because the circle also has reflections; in fact, it's impossible to have a finite figure with infinitely many rotation symmetries but no reflection symmetries.)

Task 47. *You will be given a worksheet of Japanese crests (symbols used to identify Japanese noblemen and samurai). Find the symmetry type (C_n or D_n, for some n) of each crest.*

How would we explain to someone else how to find the symmetry type of a finite figure? We could tell them to list all the possible symmetries, but that's more work than they actually need to do. We want to find a method for determining the symmetry type which is as quick as possible.

Task 48. *Describe a quick procedure for finding the symmetry type of a finite figure. Organize your procedure as a* flowchart *– i.e. as a series of questions with simple answers, with the response to each question determining the next question in the series.*

Extra Credit Task 9. *If you superimpose two finite figures with known symmetry types, what is the symmetry type of the resulting figure? How does it depend on the symmetry types of the original figures? How does it depend on how you superimpose them? You will receive partial credit for giving examples and making some reasonable conjectures; and full credit if you can prove your conjectures.*

Extra Credit Task 10. *Design several interesting finite figures with different symmetry types. The credit given will depend on how interesting your designs are, and how well they are executed.*

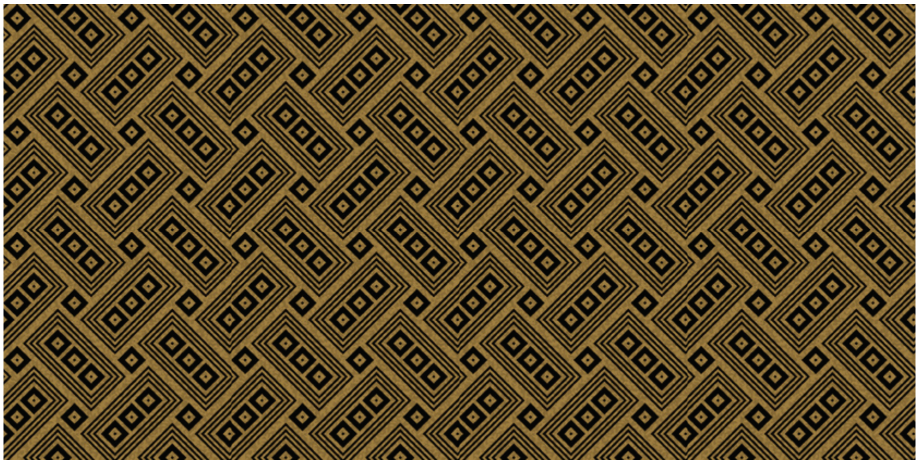


Figure 22: A piece of an infinite pattern

3 Symmetries of Infinite Patterns

In the last section, we looked at *finite* figures, so every symmetry had to fix a point. Now we are going to look at *infinite* patterns. Of course, we can't really draw an infinite pattern, so in practice we look at a representative part of the pattern, as though we were looking at the pattern through a window. As the pattern is moved in one direction, some of it will move out of view, and other parts will move into view. Figure 22 shows part of such a pattern; if we moved our "window" either right and left or up and down, we would see the pattern repeat.

This means that we have a new kind of motion to deal with, where we actually move our figure some distance in a particular direction. These motions are called *translations*.

3.1 Translations

Task 49. *1. On a piece of paper, draw a simple doodle and label three points on your doodle A, B, and C.*

2. Place a piece of tracing paper on top of the first piece paper, matching up the edges of both pieces. Trace your doodle onto the top sheet, including points A, B, C.

3. Slide the top sheet without turning it*, so that the edges of the two sheets remain parallel. Find a configuration where you can see the original doodle through the top sheet, but it does not overlap the first copy. Make a second copy of the doodle in the new position, labeling the points corresponding to A, B, C by A', B' and C'. Congratulations! You have just translated doodle ABC to doodle $A'B'C'$.*

4. On the tracing paper, draw line segments $\overline{AA'}$, $\overline{BB'}$ and $\overline{CC'}$.

5. Draw the line segments $\overline{A'B'}$ and $\overline{AB}$ to complete the quadrilateral (i.e., 4-sided figure) $AA'B'B$. What do you notice about the opposite sides of this quadrilateral?

6. Also complete the quadrilaterals $AA'CC'$ and $BB'CC'$. Do these have the same property you noticed about $AA'BB'$?

Now we can give the formal definition of a translation.

Definition 9. *A* **translation** *moves every point of the plane the same distance, and in the same direction. Formally, if the motion moves points P and Q to P' and Q', respectively, then the quadrilateral $PP'Q'Q$ is a* **parallelogram**, *as shown on the left in Figure 23. This means that $\overline{PP'}$ and $\overline{QQ'}$ are parallel and have the same length, as are $\overline{PQ}$ and $\overline{P'Q'}$. We commonly represent a translation by an arrow, where the length of the arrow gives the distance of the translation, and the direction of the arrow gives the direction, as shown on the right in Figure 23. This arrow is called the* **translation vector**.

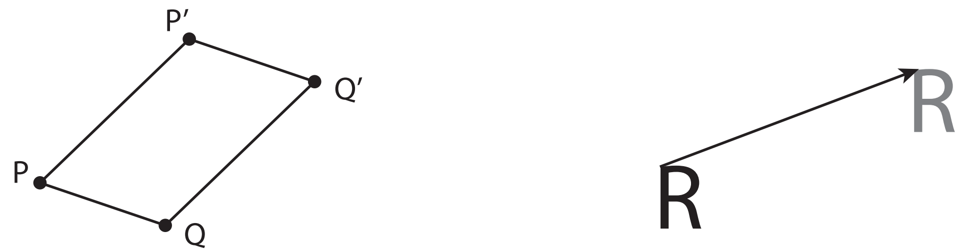


Figure 23: Translations

Task 50. *Restate this definition using your own words and possibly a picture.*

Other properties of parallelograms (aside from the fact that the opposite sides are parallel and have the same length) are that the opposite angles are the same size, and any two neighboring angles add up to 180°.

Task 51. *For this task, use the pictures in Figure 24.*

1. *What are the angles of the parallelogram $ABCD$ on the left in Figure 24? What do you notice about opposite angles? Neighboring angles?*
2. *Consider the three points A, B and C on the right in Figure 24. Find and draw the point D such that $ABCD$ is a parallelogram. Describe your method.*

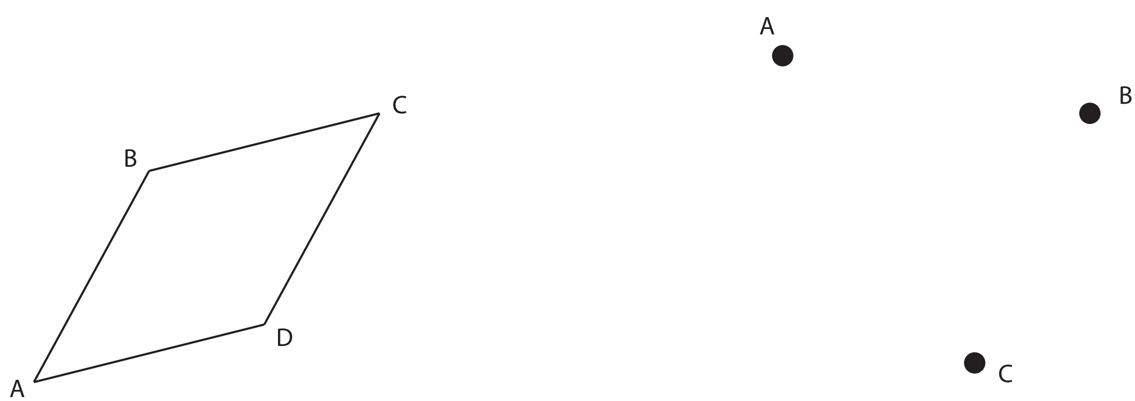


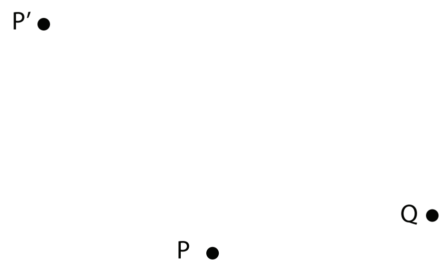
Figure 24: Figures for Task 51

Task 52. *Do the Worksheet on Translations.*

Task 53. *How can you tell if two line segments $\overline{AB}$ and $\overline{CD}$ are translations of each other? Give a list of necessary criteria.*

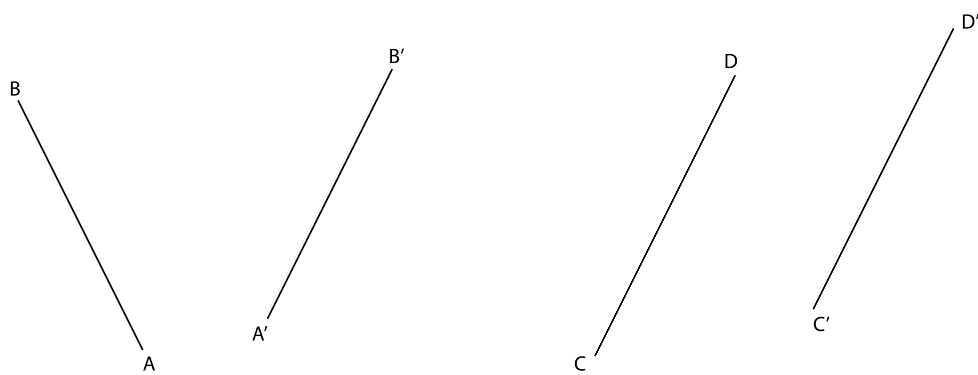
Worksheet on Translations

1. In the picture below, find the translation vector that takes P to P' (Easy!). Find the image Q' of Q under this translation. Use a ruler and protractor to locate Q' as accurately as possible. [Hint: Task 51.]



Write a few sentences describing your method.

2. Consider the two pairs of line segments below.



(a) Is there a translation which takes $\overline{AB}$ to $\overline{A'B'}$? If yes, draw the translation vector; if not, explain why not.

(b) Is there a translation which takes $\overline{CD}$ to $\overline{C'D'}$? If yes, draw the translation vector; if not, explain why not.

3. Use a ruler and protractor to accurately translate the quadrilateral below by the given vector.

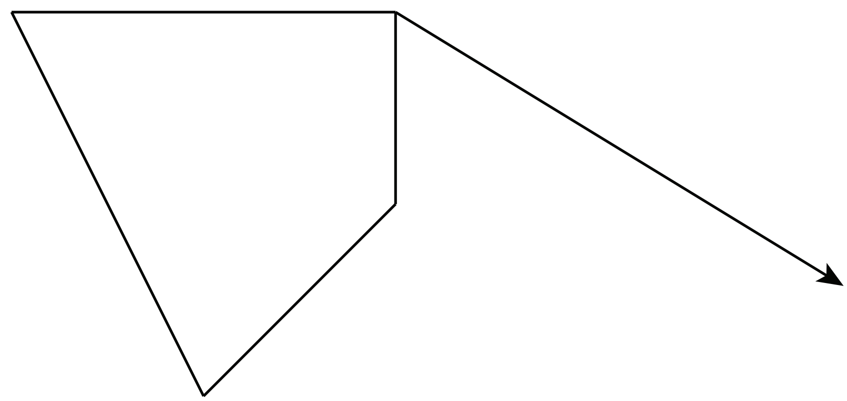


Figure 25: Equivalent rotocenters and mirror lines are marked with the same color. The pattern on the left has (up to equivalence) one rotation and one vertical reflection; the pattern on the right has two of each.

3.2 A First Look at Strip Patterns

We will begin by looking at patterns which only have translation symmetry in one direction – these are called *strip*, or *frieze*, patterns. As with finite figures, we will begin with some exploration. It will help to first define a notion of *equivalent* symmetries.

Definition 10. *Two rotation symmetries of a pattern are* **equivalent** *if they are rotations by the same angle (in the same direction), and there is another symmetry of the pattern which takes the rotocenter of the first rotation to the rotocenter of the second rotation.*

Two reflection symmetries of a pattern are **equivalent** *if there is another symmetry of the pattern which takes the mirror of the first reflection to the mirror of the second reflection.*

Some examples are shown in Figure 25. In the strip pattern on the left, all the rotation symmetries are equivalent and all the reflection symmetries are equivalent. Note that two adjacent mirror lines are equivalent by rotation around the rotocenter between them, and two adjacent rotocenters are equivalent by reflection in the vertical mirror line between them. In the strip pattern on the right, there are two inequivalent rotation symmetries and two inequivalent vertical reflection symmetries (and a horizontal reflection symmetry, whose mirror line is not shown).

Task 54. *You have been provided with a set of Native American strip patterns.*

1. *On each pattern, mark the translation vector for the smallest translation symmetry, the centers of rotation for all the rotation symmetries, and the mirror lines for all the reflection symmetries.*
2. *For each pattern, are there any symmetries which are not just a single rotation, reflection or translation? If so, describe them as a combination of rotations, reflections and translations.*
3. *How many inequivalent rotation symmetries does each pattern have? How many inequivalent reflection symmetries?*
4. *Group the patterns according to their symmetries. What are the defining characteristic of each group?*
5. *Find another group, and compare your classifications. Develop a classification that you all agree on.*
6. *Share your classification with the class.*

To decide whether the classification you've found is complete, we need to once again begin by classifying the rigid motions and seeing how they interact. But now we are not restricted to motions which fix a point.

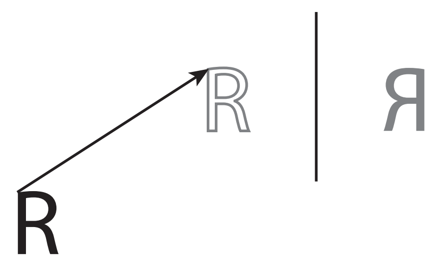
3.3 Glide Reflections

We now have three kinds of rigid motions: rotations, reflections and translations. Are there any other rigid motions? We've already seen that if we combine two rigid motions, we'll get a new rigid motion. But we've also seen that combining two rotations around the same point just gives another rotation (Theorem 3), combining a rotation with a reflection across a line through the rotocenter gives another reflection (Theorem 4), and combining two reflections gives a rotation around the point where the mirror lines intersect (Theorem 5). So maybe other combinations of rigid motions will just give one of the three we already have.

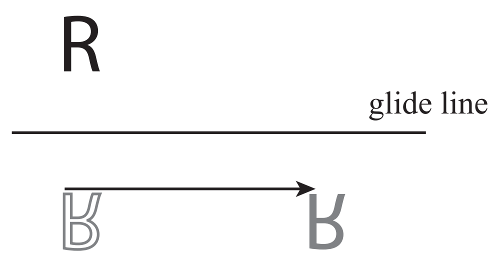
Task 55. *Draw some examples to convince yourself that if we combine two translations, the result is another translation. Given the translation vectors (i.e., arrows) for the first two translations, how do you find the translation vector for the new translation?*

But what if we combine translations with another kind of motion – in particular, with a reflection. Is the result always a translation, rotation or reflection? No!

Task 56. *Consider the combination of a translation and reflection shown below (the outlined* R *shows the result of the translation, the gray* Я *is the result of following it with the reflection). Explain why this cannot be the result of a single rotation, reflection or translation.*



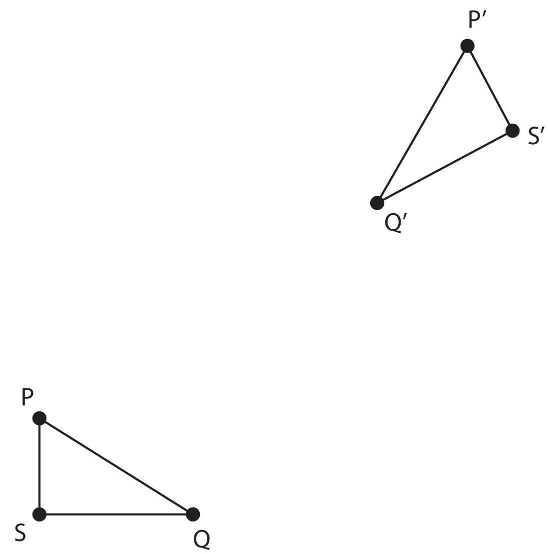
We're going to define a new kind of motion called a *glide reflection* to deal with this. A glide reflection is the result of first doing a reflection, and then doing a translation *parallel* to the mirror line. The mirror line of the reflection is called the *glide line* of the glide reflection. An example is shown below – the outlined Я is the image of the initial R under the reflection, and the gray Я is the final image.



Why do we bother to define a glide reflection? Why not just say that some motions are combinations of translations and reflections? The reason is that we want to think of each rigid motion as a unique thing, with a single unambiguous description. But if we describe motions as combinations, then there are many different ways to describe any given motion, which becomes confusing. If each rigid motion can be described in many ways, it is much more difficult to tell whether two of them are the really the same. However, if every rigid motion has a unique "standard" description, these comparisons are much easier.

The next task will show you how to find the glide line and translation vector for a glide reflection.

Task 57. *We want to find the glide reflection which takes $\triangle PQR$ to $\triangle P'Q'S'$ in the picture below. Begin by copying this picture into your journal.*



1. *Draw the line segments $\overline{PP'}$ and $\overline{QQ'}$.*
2. *Find and mark the midpoints of the two line segments.*
3. *Draw the line g which goes through the two midpoints.*
4. *Reflect $\triangle PQS$ across g, to get $\triangle P^*Q^*S^*$.*
5. *Draw the translation vector $\mathbf{v}$ from P^* to P'. Observe that $\mathbf{v}$ is parallel to g.*

Therefore, $\triangle P'Q'S'$ is the glide reflection of $\triangle PQS$, with glide line g and translation vector $\mathbf{v}$ parallel to g.

We should note that if a rigid motion is a glide reflection, there is only *one* glide reflection that will work. In other words, while we may be able to get the same motion with different combinations of translations and reflections, there is only one such combination where the translation is parallel to the line of reflection. This means that we can talk about *the* glide reflection for a particular motion.

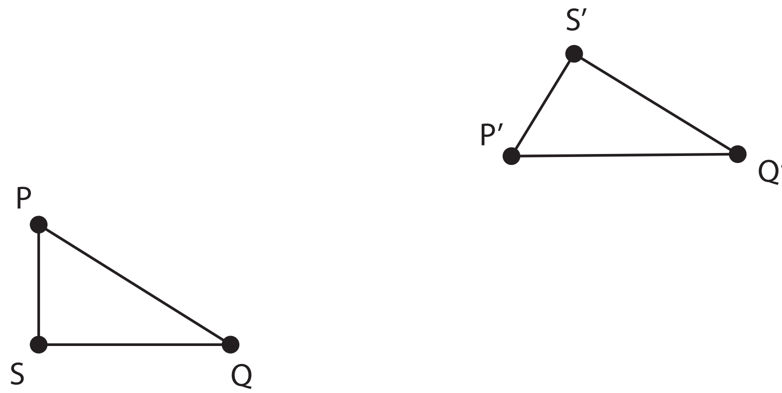
Task 58. *Do the Worksheet on Glide Reflections.*

Task 59. *Look again at the strip patterns from Task 54.*

1. *Which of the strip patterns have a glide reflection symmetry? This symmetry may be the result of combining a reflection symmetry and a translation symmetry.*
2. *In which cases does the strip pattern have a glide reflection symmetry which is* **not** *the combination of a reflection symmetry and a translation symmetry? (i.e., the reflection and the translation are not themselves symmetries of the patterns, but the combination is.)*
3. *Are there any symmetries of the strip patterns that cannot be written as a single rotation, reflection, translation or glide reflection?*

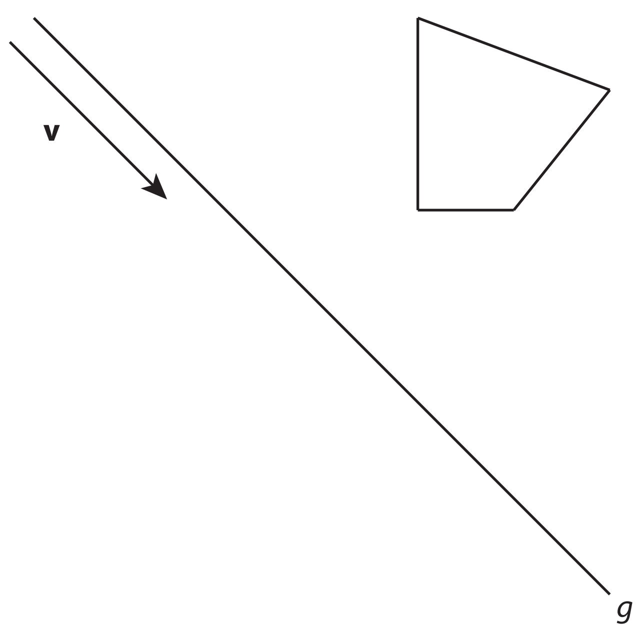
Worksheet on Glide Reflections

1. Consider the pair of triangles below.

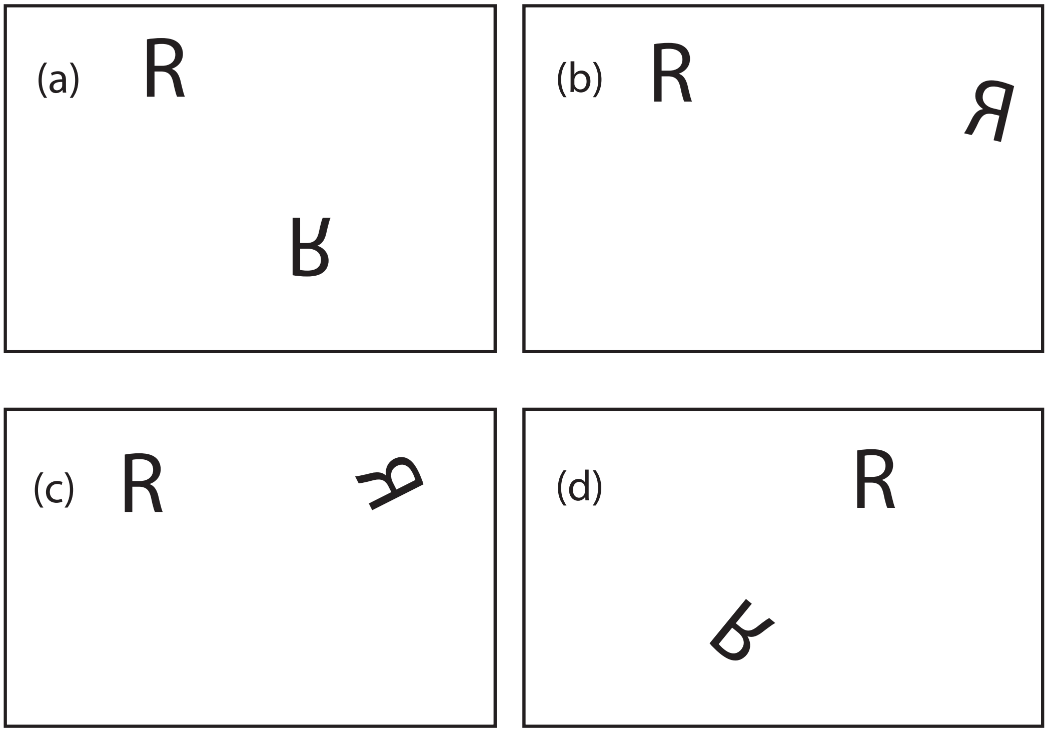


   Find and draw the glide line and translation vector for the glide reflection which takes $\triangle PQS$ to $\triangle P'Q'S'$.

2. Glide reflect the quadrilateral below, using the glide line g and the translation vector $\mathbf{v}$.



3. In each of the figures below, the R has been reflected or glide reflected to the Я. In each case, determine whether the motion is a reflection or glide reflection, find the mirror or glide line and (if it's a glide reflection), find the glide distance.



3.4 Classification of Rigid Motions

So now we have four kinds of rigid motions: rotations, reflections, translations and glide reflections. Is every rigid motion on of these four types? In this section, we will show that the answer is YES! To begin with, let's convince ourselves that any rigid motion can be classified as one of these four by looking at some examples.

Task 60. *Do the Worksheet on Rigid Motions.*

Task 61. *For each type of motion, say whether it sends an* R *to another* R *or to a* Я.

1. *Rotation*
2. *Translation*
3. *Reflection*
4. *Glide Reflection*

In fact, *every* rigid motion needs to send an R to another R or to a Я. To see this, we will prove a generalization of Theorem 6, showing that every rigid motion is a combination of translations, rotations and reflections.

Theorem 8. *Any rigid motion is either*

1. *the result of combining a translation and a rotation, or*
2. *the result of combining a translation and a reflection.*

Intuitive Proof: Let W be the rigid motion. By Theorem 2, we just need to consider three points P, Q, S. We can move P to $W(P)$ by a translation, and then move Q and S into place (fixing P) as we did in Theorem 6. □

Proof. We begin by picking an arbitrary rigid motion W, and three points P, Q and S which do not all lie on the same line. By Theorem 2, it is enough to show that we can find a combination of a translation and a rotation, or of a translation, a rotation and a reflection, which sends P to $W(P)$, Q to $W(Q)$ and S to $W(S)$. Figure 26 shows an example of points P, Q and S and their images under W.

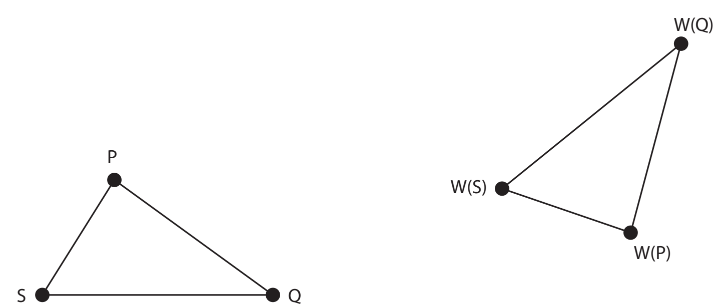


Figure 26: Points P, Q and$S$ and their images under W.

Our first step is to find the translation T which moves P to $W(P)$ (so $T(P) = W(P)$). If we move all three points P, Q and S by T, we get the gray triangle in Figure 27.

Now we need to find the motion which fixes $W(P)$, moves $T(Q)$ to $W(Q)$ and moves $T(S)$ to $W(S)$. But by Theorem 6, this motion is either a rotation around $W(P)$, or a reflection in a line through $W(P)$. So when we combine this motion with T, we get either the result of combining a translation and a rotation, or the result of combining a translation and a reflection. This combination moves P, Q and S to $W(P)$, $W(Q)$ and $W(S)$, so by Theorem 2 it must be the same motion as W. □

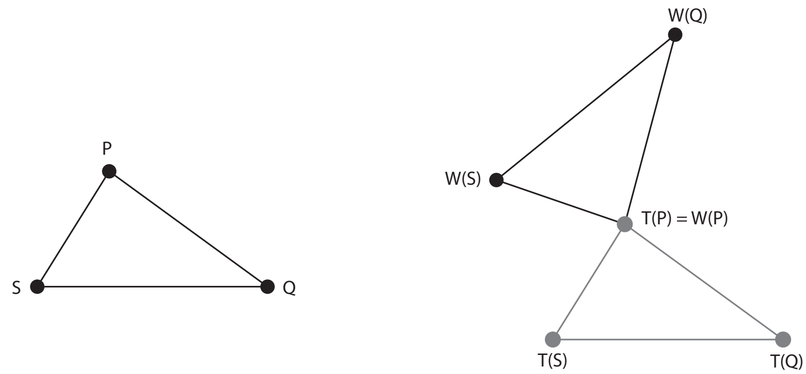


Figure 27: Points P, Q and$S$ and their images under T and W.

So any motion is either a combination of a translation and a rotation, which will send an R to another R or it's a combination of a translation and a reflection, which will send an R to a Я.

Task 62. *Assume that a rigid motion W takes an* R *to another* R*.*

1. *Explain how to determine whether W is a translation or rotation.*
2. *If W is a translation, how do you find the translation vector?*
3. *If W is a rotation, how do you find the center and angle of rotation?*

Task 62 proves that any rigid motion that takes an R to another R is a translation or a rotation.

Task 63. *Assume that a rigid motion W takes an* R *to a* Я*.*

1. *Explain how to determine whether W is a reflection or glide reflection.*
2. *If W is a reflection, how do you find the mirror line?*
3. *If W is a glide reflection, how do you find the glide line and glide amount?*

Similarly to Task 62, Task 63 proves that any rigid motion that takes an R to a Я is either a reflection or a glide reflection. (We've relegated a technical point to Extra Credit Task 11.)

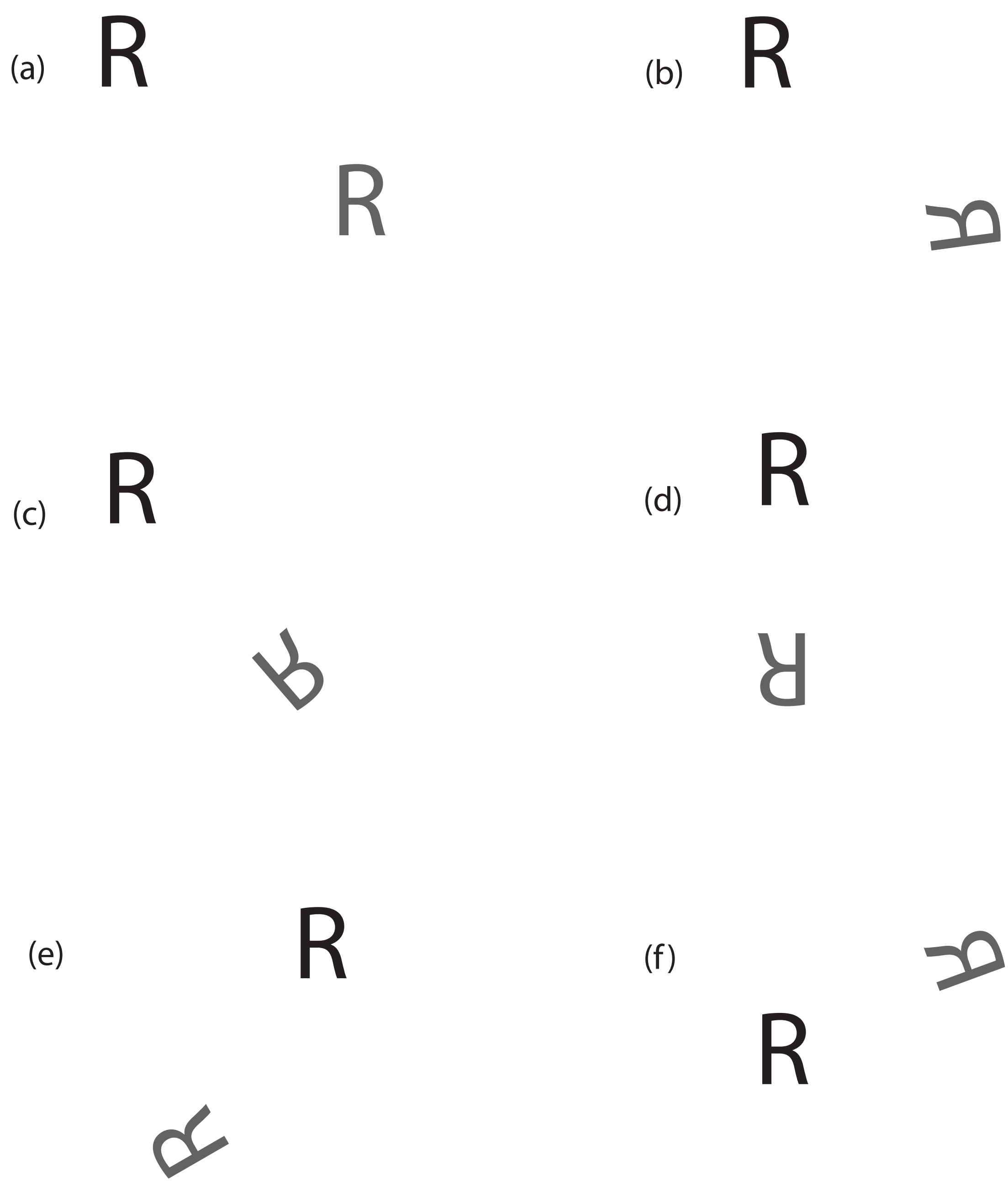
Extra Credit Task 11. *Assume that P and P' are points in the plane, and g is a line which passes through the midpoint of $\overline{PP'}$. Let Q be the result of reflecting P across g. Prove that the translation vector from Q to P' is parallel to g. [Hint: You'll need to remember some facts about similar triangles from high school geometry.]*

If we combine the results of Tasks 62 and 63, we obtain our final and complete classification of rigid motions of the plane!

Classification of Rigid Motions of the Plane. *Every rigid motion of the plane is a rotation, reflection, translation or glide reflection.*

Worksheet on Rigid Motions

Identify each of the motions below as a rotation, translation, reflection or glide reflection. For each motion, you are given an R (in black) and its image (in gray). If it is a rotation, find the center and angle of rotation; if it is a reflection, find the mirror line; if it is a translation, find the translation vector; and if it is a glide reflection, find the glide line and the glide amount.



3.5 Combining Rigid Motions

When we classified the symmetry types of finite figures, we needed to analyze what happened when we combined rotations and reflections. Now we want to investigate what happens when we combine all four kinds of rigid motions. However, for our purposes, we mostly care about what *kind* of rigid motion results from combining two rigid motions. We've already done this for a few cases:

- If we combine a translation with a translation, the result is a translation (Task 55).
- If we combine a rotation with another rotation with the same center of rotation, the result is a rotation (Theorem 3).
- If we combine a rotation with a reflection whose mirror line goes through the center of rotation, the result is a reflection (Theorem 4).
- If we combine a reflection with another reflection, and their mirror lines meet in a point P, the result is a rotation around P (Theorem 5).

Now we want to see what happens in all the other cases.

Task 64. *In each of the following cases, state what kind of rigid motion (rotation, reflection, translation or glide reflection) is the result of combining the two given rigid motions. In each case, sketch an example to illustrate your answer.*

1. *Two rotations with* **different** *centers of rotation.* **Note:** *There are some situations where the combination is a translation – when does this happen?*
2. *Two reflections whose mirror lines are* **parallel***.*
3. *A rotation and a reflection, where the mirror line does* **not** *go through the center of rotation.*
4. *A translation and a rotation.*
5. *A translation and a reflection, where the translation vector is perpendicular to the mirror line.*
6. *A translation and a reflection, where the translation vector is* **not** *perpendicular to the mirror line.*
7. *A translation and a glide reflection.* **Note:** *There are some situations where the combination is a reflection – when does this happen?*
8. *A glide reflection and a rotation.* **Note:** *There are some situations where the combination is a reflection, but they are difficult to describe precisely.*
9. *A glide reflection and a reflection, where the glide line and the mirror line are parallel.*
10. *A glide reflection and a reflection, where the glide line and the mirror line are* **not** *parallel (i.e., they cross at some point).*
11. *Two glide reflections whose glide lines are parallel.*
12. *Two glide reflections whose glide lines are* **not** *parallel (i.e., they cross at some point).*

Task 65. *Draw an example to illustrate that the combination of a rotation by* $180°$ *with a glide reflection whose glide line goes through the center of rotation is a reflection in a mirror perpendicular to the glide line.*

Extra Credit Task 12. *Give a complete description of when the combination of a rotation and a glide reflection is a reflection.*

3.6 Strip (or Frieze) Patterns

Now we return to our study of strip patterns. Let's formally state our definition of a strip pattern.

Definition 11. *A* **strip pattern** *(or* **frieze pattern***) is a pattern which has translation symmetries in* **one** *direction.*

These patterns are commonly used in textiles, pottery and architectural ornament. In our examples, we will usually draw horizontal strips, but in practice they can extend in any direction. You examined several examples in Task 54.

An important thing to keep in mind about a strip pattern is that any symmetry of the pattern must move the *entire* pattern – it can't move just one part of the design without moving the whole strip.

Task 66. *Explain why strip patterns can only have* 180° *rotation symmetries.*

You will notice that strip patterns can have two kinds of reflections: reflections in vertical mirrors and reflections in a horizontal mirror down the center of the strip. Reflections through any other mirror would change the direction of the strip, as in Figure 28. However, the glide reflections are always horizontal, since the strip can only be shifted horizontally and still look the same.

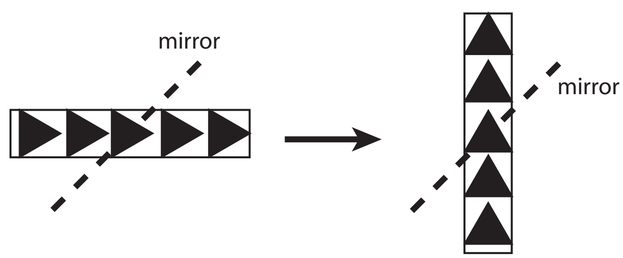


Figure 28: Reflecting a strip pattern in a diagonal mirror changes its direction

So there are five possible types of symmetries a (horizontal) strip can have: horizontal translation (which it must have, by definition of a strip pattern), 180° rotation, reflection in a vertical mirror, reflection in a horizontal mirror, and glide reflection with a horizontal glide line. Two strip patterns have the same symmetry type if they have the same collection of symmetries.

Task 67. *Return to the set of examples in Task 54 and list which of the five types of symmetries each pattern has. Do you get the same classification you found before?*

We'd like to know how many symmetry types are possible. We know that every strip pattern has to have a translation, but which combinations of the other types are possible? There are four other types of symmetries, so there are $2^4 = 16$ possible combinations of them.

Task 68. *List all 16 possible combinations of symmetries for a strip pattern (include translation in each case). For convenience, use* t *for translation,* r *for a* 180° *rotation,* v *for reflection in a vertical mirror,* h *for reflection in a horizontal mirror, and* g *for a glide reflection. So the collection consisting of translation, rotation and vertical reflection would be listed as "*t, r, v*".*

Task 69. *Take photographs of strip patterns around campus. Try to find patterns with as many of the 16 combinations of symmetries as possible. For those combinations you don't find, try to design patterns with each combination of symmetries, and no others. Can you find or design patterns for each of the 16 combinations?*

You should have found that there were some combinations of symmetries you couldn't get – whenever you tried, some other symmetries kept appearing. We want to find some rules that will let us determine which combinations of symmetries are impossible. These rules will be based on the idea that a product of two symmetries of a pattern is also a symmetry. We give one such rule as an example:

Rule 1. *If a strip pattern has a horizontal reflection symmetry, then it has a glide reflection symmetry.*

Proof. Every strip pattern has a translation symmetry. If we combine a horizontal translation with the reflection in a horizontal mirror, the result is a glide reflection. Therefore, the strip pattern must also have a glide reflection symmetry. □

Using this rule, we can eliminate any combination of symmetries which includes translation and horizontal reflection, but not glide reflection. Essentially, this rule is based on a "multiplication fact": namely, the "product" of a translation and a horizontal reflection is a glide reflection.

Task 70. *Complete the attached "multiplication table" for the five possible symmetries of a strip pattern. Be sure to mention any special cases (for example, combining two vertical reflections gives a translation – but if they had the same mirror line, the result is the identity).*

Task 71. *Use the multiplication table to find as many other rules as you can, until you can eliminate all the combinations from Task 68 that you cannot find examples for.*

Task 72. *Complete the attached Worksheet on Strip Patterns. For each collection of symmetries, either draw an example of a strip pattern with exactly those symmetries (and no others), or state the rule that makes it impossible to find a strip pattern with exactly those symmetries.*

In the end, there are only 7 possible combinations of symmetries, the 7 frieze groups. It is useful to have a shorthand notation to refer to them (rather than talking about "the one with translations and rotations"). We will look at a couple of common notations.

Princeton mathematician John Conway has developed a terminology based on footprint patterns: hop, step, jump, sidle, spinning hop, spinning jump, spinning sidle.

Task 73. *Figure out which pattern belongs with which term, and draw a footprint pattern to illustrate it.*

Another terminology is the $pxyz$ notation. Each begins with p (representing the translation). The second, third and fourth symbols describe vertical reflections, horizontal reflections or glide reflections, and 180° rotations, respectively, as follows

- x is m (for mirror) if there is a vertical reflection, and 1 otherwise.
- y is m if there is a horizontal reflection, a if there is a glide reflection but no horizontal reflection, and 1 otherwise.
- z is 2 if there is a 180° rotation symmetry, and 1 otherwise.

Task 74. *What are the possible patterns in pxyz notation? How do they correspond to the Conway notation?*

Task 75. *Classify the strip patterns from Task 54 using the Conway and pxyz notations.*

Task 76. *Identifying all the symmetries in a pattern can be tedious. Develop a flowchart that allows the user to determine the symmetry type of a strip pattern in as few steps as possible.*

Task 77. *You will be given a second collection of Native American strip patterns. Use your flowchart to classify the patterns. Give the symmetry type in both the Conway and pxyz notations.*

Extra Credit Task 13. *Create an interesting design that incorporates all 7 strip patterns. (A simple collection of 7 strips drawn side by side is* **not** *interesting.)*

“Multiplication Table” for Task 70

Below is a “multiplication table” for the symmetries of a horizontal strip pattern (so the translation symmetry is horizontal). Complete the table by describing the *type* of symmetry which results by combining the symmetries in each row and column. If there are multiple possibilities, briefly describe under what conditions each possibility occurs.

Notice that, since you are only finding the *type* of the resulting motion, the order in which two motions are combined does *not* effect the result.

	translation	180° rotation	vertical reflection	horizontal reflection	glide reflection
translation	translation			glide reflection	
180° rotation					
vertical reflection			translation (identity if same mirror)		
horizontal reflection	glide reflection				
glide reflection					

Worksheet on Strip Patterns for Task 72

For each collection of symmetries, either draw a strip patterns with *exactly* those symmetries, or explain why it is not possible. The symmetries are abbreviated as follows:

- t = translation
- r = 180° rotation
- v = vertical reflection
- h = horizontal reflection
- g = glide reflection

t	$t,\ r$
$t,\ v$	$t,\ h$
$t,\ g$	$t,\ r,\ v$
$t,\ r,\ h$	$t,\ r,\ g$

$t,\ v,\ h$	$t,\ v,\ g$
$t,\ h,\ g$	$t,\ r,\ v,\ h$
$t,\ r,\ v,\ g$	$t,\ r,\ h,\ g$
$t,\ v,\ h,\ g$	$t,\ r,\ v,\ h,\ g$

3.7 Wallpaper Patterns

Patterns with translation symmetries in two directions (along two non-parallel lines) are called **wallpaper patterns**. These patterns appear in a wide range of art and designs, from M.C. Escher's famous prints to Islamic mosaics to (of course) wallpaper.

Task 78. *You will be given a worksheet of wallpaper patterns. Find all the symmetries in the patterns, marking the (inequivalent) rotocenters, mirror lines and glide lines. Label each rotocenter with the order of the smallest rotation symmetry around that point (i.e. the number of times the rotation needs to be repeated to come around* $360°$*). Count the number of inequivalent rotation symmetries, reflection symmetries and glide reflection symmetries in each pattern.*

Is there a limit to the number of possible wallpaper patterns? At first glance, there is no reason to think so – after all, in two-dimensional patterns, couldn't you have rotations by any amount, as we did for finite figures? No! To understand why, we need to look at tilings of the plane by regular polygons. Which regular polygons can be used to tile the plane? Figure 29 shows how to tile the plane with triangles, squares and hexagons. In fact, these are the only possibilities!

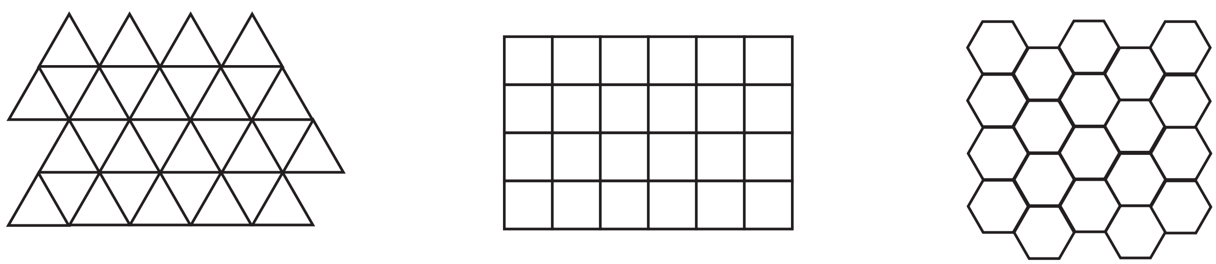


Figure 29: Tiling the plane with regular polygons

Theorem 9. *The only regular polygons we can use to tile the plane are the equilateral triangle, the square and the regular hexagon.*

Proof. Remember that a regular polygon with n sides has an exterior angle of $360/n$, and so an interior angle of $180-360/n = (n-2)180/n$. To tile the plane, some number of these polygons need to fit together around each vertex – i.e. there is some integer k such that $k(n-2)180/n = 360$, so $k = 2n/(n-2) = 2+4/(n-2)$. So once $n > 6$, $0 < 4/(n-2) < 1$, so k is not an integer. Also, for $n = 5$, $k = 2 + 4/3$, which is not an integer. So the only possible values of n are 3 ($k = 6$), 4 ($k = 4$) and 6 ($k = 3$). □

The restrictions on the ways to tile a plane with regular polygons leads to the so-called crystallographic restriction for wallpaper patterns: the only possible rotation symmetries in a wallpaper pattern are rotations by 1/2 turn, 1/3 turn, 1/4 turn or 1/6 turn. Beginning with this restriction, it is possible to determine the possible combinations of other symmetries, as we did for strip patterns – for example, mirror lines can only intersect at angles of $90°$, $60°$, $45°$ or $30°$. It turns out that there are 17 possible wallpaper patterns. The details of the proof are too intricate to go into here (though the ideas are essentially the same as for strip patterns). For those interested, a complete proof can be found in [Cro]. Table 2 (from [Sch]) lists the 17 patterns (using notation based on that used by crystallographers for three-dimensional crystalline structures), along with some identifying characteristics.

Task 79. *Use Table 2 to classify the patterns from Task 78.*

Task 80. *Use the information in Table 2 to develop a flowchart for quickly identifying wallpaper patterns, as you did for strip patterns in Task 76.*

Type	Highest order of rotation	Mirror lines	Has glide lines which are not mirror lines	Other characteristics
$p1$	1	no	no	
pm	1	yes	no	
pg	1	no	yes	
cm	1	yes	yes	
$p2$	2	no	no	
pmm	2	yes	no	
pmg	2	yes	yes	mirror lines do not intersect
pgg	2	no	yes	
cmm	2	yes	yes	mirror lines do intersect
$p4$	4	no	no	
$p4m$	4	yes	yes	Order 4 rotocenters are on mirror lines
$p4g$	4	yes	yes	Order 4 rotocenters are not on mirror lines
$p3$	3	no	no	
$p3m1$	3	yes	yes	All order 3 rotocenters are on mirror lines
$p31m$	3	yes	yes	Not all order 3 rotocenters are on mirror lines
$p6$	6	no	no	
$p6m$	6	yes	yes	

Table 2: Identifying characteristics for wallpaper patterns

Task 81. *Take photographs of wallpaper patterns you can find on campus or in the neighborhood. How many different symmetry types can you find? Classify each pattern you find. You should have at least 10 patterns, with at least 3 different types, and at least one pair of distinct patterns that have the same symmetry type.*

4 Acknowledgements

I am grateful to Gwen Fisher for reading these notes, and for suggesting many excellent improvements. I am also grateful for her permission to use her excellent diagrams for folding regular polygons and paper snowflakes, and for providing background on real snowflakes.

References

[Cro] Crowe, D. *Symmetry, Rigid Motions, and Patterns.* Arlington, Mass.: COMAP, 1986.

[Sch] Schattschneider, D. "The Plane Symmetry Groups: Their Recognition and Notation," *The American Mathematical Monthly*, Vol. 85, No. 6, 1978, pp. 439–450