Math 550 Project: Curvature in space

Blake Mellor, Loyola Marymount University

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1 Introduction

The purpose of this project is to provide a brief introduction to some of the fundamental notions of differential geometry. Differential Geometry is the use of analysis to study the geometry of curves and surfaces embedded in larger spaces; essentially, it is an extension of multivariable calculus. In this project, you will develop the notion of the *curvature* of a surface in space – a concept which is important in the study of general relativity, among other applications.

The project will begin with the simpler study of curves in space, and use the tools we develop for curves to study surfaces. This is a project, not a textbook – you will be expected to work through many of the calculations and proofs for yourself. However, you should feel free to come to me for assistance at any time.

You will do all the Exercises in this project and turn them in according to the schedule below.

Date	**Assignment**
Friday, September 7	Section 2
Friday, September 21	Section 3
Friday, October 5	Sections 4 and 5
Friday, October 19	Sections 6 and 7
Friday, November 2	Sections 8 and 9
Friday, November 16	Section 10 and Exercise 42
Friday, November 30	Section 11

2 Curves in Space

Definition 1. *A* ***smooth, parametrized curve*** *is a mapping $\alpha : (a, b) \to \mathbb{R}^n$, $-\infty \leq a < b \leq \infty$, such that $\alpha(t)$ has derivatives of all orders. The* ***trace*** *of α is its image as a set of points in $\mathbb{R}^n$. (For us, n will be 2 or 3.)*

The ***tangent vector*** *to α at $\alpha(t_0)$ is the derivative $\alpha'(t_0) = \frac{d\alpha}{dt}(t_0)$. The* ***speed*** *of α at $\alpha(t_0)$ is $\|\alpha'(t_0)\| = \sqrt{\alpha'(t_0) \cdot \alpha'(t_0)}$.*

A smooth curve α is ***regular*** *if $\frac{d\alpha}{dt} \neq 0$ for all $t \in (a, b)$ (i.e. the speed is never 0).*

You have seen many examples of parametrized curves in multivariable calculus.

Exercise 1.
1. *Give two different smooth curves in $\mathbb{R}^2$ whose traces are the circle of radius 2 centered at the point $(2, 3)$.*
2. *Draw the trace of the curve $\alpha(t) = (t, t^{2/3})$. Is this a smooth curve?*
3. *Find a smooth curve with the same trace as in the last question. Is this curve regular?*
4. *Describe the trace of the curve $\alpha(t) = (2 \cos t, 2 \sin t, 3t)$.*

Exercise 2. *Why do we restrict ourselves to regular curves? What features does this avoid?*

Exercise 3. *Show that the trace of a curve $\alpha : (a, b) \to \mathbb{R}^3$ lies on the surface of a sphere centered at the origin (i.e. $\|\alpha(t)\|$ is constant) if and only if $\alpha(t) \perp \alpha'(t)$ for all $t \in (a, b)$.*

As you have seen, a given curve (i.e. trace of a curve) may have many different parametrizations. If $\alpha : (a, b) \to \mathbb{R}^n$ is a regular smooth curve, a *reparametrization* of α is a function $g : (c, d) \to (a, b)$ which is smooth with a smooth inverse. The reparametrized curve is $\beta = \alpha \circ g$. If we are primarily interested in the geometrical properties of the trace of the curve, we want to study measurements which are independent of the parametrization. The first of these is the *arc length* of the curve.

Definition 2. *Let $\alpha : (a, b) \to \mathbb{R}^n$ be a regular, parametrized, smooth curve, with $a < a_0 < b$. The* ***arc-length function*** *$s : (a, b) \to \mathbb{R}$ giving the distance along the curve from a_0 to $t \in (a, b)$ is defined by:*

$$s(t) = \int_{a_0}^{t} \|\alpha'(r)\| dr$$

Observe that if $a < t < a_0$, then the arc-length $s(t)$ will be negative.

Exercise 4. *Prove that the arc-length between two points on a curve is independent of the parametrization. [Hint: chain rule]*

Exercise 5. *Compute the arc length of the cycloid $\xi(t) = (t + \sin t, 1 - \cos t)$, for $-\pi \le t \le \pi$. Sketch the curve. A* ***cycloid*** *is the arc swept out by a fixed point on a circle as the circle rolls along a straight line.*

We can use the arc-length function to define a canonical parametrization for a curve, called the *arc-length parametrization*, as follows. Begin with a regular smooth curve $\alpha : (a, b) \to \mathbb{R}^n$. Pick a point $a_0 \in (a, b)$, and let $s(t)$ denote the arc-length function giving the distance from $\alpha(a_0)$ to $\alpha(t)$.

Exercise 6. *Prove that $s(t)$ is invertible. [Hint: you will need the fact that α is regular.]*

Now let (c, d) be the image of (a, b) under s, and define $\beta = \alpha \circ s^{-1}$. β is the arc-length parametrization for the trace of α, and has several nice properties.

Exercise 7.

1. *Show that the arc-length from $\beta(x)$ to $\beta(y)$ is $y - x$.*
2. *Show that β is* unit-speed*, meaning that $\|\beta'(t)\| = 1$ for all $t \in (c, d)$.*

Exercise 8. *Reparametrize each of the following curves with respect to arc length.*

1. *The helix $\alpha(t) = (a \cos t, a \sin t, bt)$.*
2. *$\alpha(t) = (e^t \cos t, e^t \sin t, e^t)$.*

In practice, it is often difficult, if not impossible, to reparametrize a curve with respect to arc length. Doing so requires being able to first compute the arc-length function $s(t)$, and then compute its inverse. There are relatively few curves for which we can find an explicit algebraic formula for the arc-length parametrization. Nonetheless, knowing that such reparametrizations *exist*, even if we can't find them explicitly, is often very useful theoretically. We usually begin our study of any kinds of curves by assuming they are parametrized with respect to arc length, which allows us to use the many nice properties of such curves, such as the one given in Exercise 9.

Exercise 9. *Prove that if $\alpha(t)$ is a unit-speed curve, then $\alpha'(t) \perp \alpha''(t)$ for all t.*

3 Properties of Curves in Space

We will make substantial use of the *cross-product* of two vectors in $\mathbb{R}^3$, so first let's review the definition and properties of the cross product.

Definition 3. *Let $\mathbf{u} = (u_1, u_2, u_3)$ and $\mathbf{v} = (v_1, v_2, v_3)$ be two vectors in $\mathbb{R}^3$. The* ***cross-product*** *$\mathbf{u} \times \mathbf{v}$ is defined by:*

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & u_1 & v_1 \\ \mathbf{j} & u_2 & v_2 \\ \mathbf{k} & u_3 & v_3 \end{vmatrix} = (u_2 v_3 - u_3 v_2, u_3 v_1 - u_1 v_3, u_1 v_2 - u_2 v_1)$$

Theorem 1. *Recall the following properties of the cross-product:*

1. *$\mathbf{u} \times \mathbf{v} = -(\mathbf{v} \times \mathbf{u})$.*
2. *$\mathbf{u} \times \mathbf{u} = \mathbf{0}$.*
3. *$\mathbf{u} \times (a\mathbf{v} + b\mathbf{w}) = a(\mathbf{u} \times \mathbf{v}) + b(\mathbf{u} \times \mathbf{w})$.*
4. *$(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w} = \begin{vmatrix} \mathbf{w} & \mathbf{u} & \mathbf{v} \end{vmatrix}$.*
5. *If $\mathbf{u} \times \mathbf{v} \neq \mathbf{0}$, then $\mathbf{u} \perp (\mathbf{u} \times \mathbf{v})$ and $\mathbf{v} \perp (\mathbf{u} \times \mathbf{v})$.*

6. $\|\mathbf{u} \times \mathbf{v}\|^2 = \|\mathbf{u}\|^2\|\mathbf{v}\|^2 - (\mathbf{u} \cdot \mathbf{v})^2 = \|\mathbf{u}\|^2\|\mathbf{v}\|^2 \sin^2\theta$, *where* θ *is the angle from* $\mathbf{u}$ *to* $\mathbf{v}$.

7. $\frac{d}{dt}(\alpha(t) \times \beta(t)) = (\alpha'(t) \times \beta(t)) + (\alpha(t) \times \beta'(t))$.

Now we can define some of the quantities we will associate with a curve, including its *curvature*.

Definition 4. *Say that* $\alpha(s)$ *is a unit-speed smooth curve in* $\mathbb{R}^3$.

- $T(s) = \alpha'(s)$ *is the* ***unit tangent vector***.
- $\kappa(s) = \|\alpha''(s)\| = \|T'(s)\|$ *is the* ***curvature*** *at* s.
- $N(s) = \alpha''(s)/\kappa(s)$ *(when* $\kappa(s) > 0$*) is the* ***unit normal vector***. *Note that* $\alpha' \perp \alpha''$ *(by Exercise 9), so* $T(s) \perp N(s)$.
- $B(s) = T(s) \times N(s)$ *is the* ***unit binormal vector***. *Note that* $\|B(s)\|^2 = \|T(s)\|^2\|N(s)\|^2 - T(s) \cdot N(s) = 1 \cdot 1 - 0 = 1$.

Note that $(T(s), N(s), B(s))$ form an orthonormal basis for $\mathbb{R}^3$ for each s. Obviously, some problems arise when $\kappa(s) = 0$. For the moment, assume $\kappa(s) > 0$.

Exercise 10. *Compute the unit tangent vector, curvature and unit normal vector at each point of a circle of radius* r *in the xy-plane, centered at the origin. What happens to the curvature as the radius increases? How does this justify the term "curvature"?*

Exercise 11. *Say that* $\alpha(s)$ *is a unit speed curve in* $\mathbb{R}^3$, *with* $\kappa(s) > 0$. *Then* $B(s)$ *is constant if and only if* $\alpha(s)$ *is planar (i.e. the trace of* α *lies in a single plane). [Hint: for the "only if" part, show that the vector* $\alpha(s) - \alpha(0)$ *is always perpendicular to the constant unit binormal vector.]*

Exercise 12. *If* $\alpha(s)$ *is a unit-speed curve, show that* $B'(s)$ *is parallel to* $N(s)$. *[Hint: Show* $B'(s)$ *is perpendicular to both* $B(s)$ *and* $T(s)$.*]*

Definition 5. *The* ***torsion*** *of* α *at* s *is the constant* $\tau(s)$ *such that* $B'(s) = -\tau(s)N(s)$.

Unlike the curvature, torsion may be either positive or negative. Note that the torsion is identically 0 if and only if the unit binormal vector is constant, and hence if and only if the curve is planar. So the torsion measures how "non-planar" the curve is.

Definition 6. *The set* $\{\kappa(s), \tau(s), T(s), N(s), B(s)\}$ *is called the* ***Frenet-Serret apparatus*** *for the curve* $\alpha(s)$.

Exercise 13. *Compute the Frenet-Serret apparatus for the following curves.*

1. $\beta(s) = ((4/5)\cos s, 1 - \sin s, (-3/5)\cos s)$. *Describe this curve.*

2. *The helix* $\alpha(t) = (a\cos t, a\sin t, bt)$. *Be careful - first you need to reparametrize with respect to arc length!*

Frenet-Serret Theorem. *If* $\alpha(s)$ *is a unit speed curve with* $\kappa(s) > 0$, *then* $T'(s) = \kappa(s)N(s)$, $N'(s) = -\kappa(s)T(s) + \tau(s)B(s)$ *and* $B'(s) = -\tau(s)N(s)$.

Exercise 14. *Prove the Frenet-Serret Theorem. Notice that two of the three equalities are by definition; only the formula for* $N'(s)$ *requires proof.*

The work we have gone to to develop the Frenet-Serret apparatus for a curve is justified by the following theorem.

Fundamental Theorem for Curves in Space. *Let* $\kappa, \tau : (a, b) \to \mathbb{R}$ *be continuous functions with* $\kappa > 0$. *Then there exists a curve* $\alpha : (a, b) \to \mathbb{R}^3$, *parametrized by arc length, with curvature* κ *and torsion* τ. *Moreover, if* $\overline{\alpha}$ *is another such curve, then it differs from* α *by a proper rigid motion. I.e. there is an orthogonal matrix* A *with determinant 1, and a vector* $\mathbf{v}$, *such that* $\overline{\alpha}(s) = A\alpha(s) + \mathbf{v}$ *for every* $s \in (a, b)$.

Proof. The proof is an exercise in the existence and uniqueness of solutions to differential equations, where the equations arise from the Frenet-Serret Theorem. See [1] for details. □

We should note that the assumption that $\kappa > 0$ is necessary for the uniqueness part of the solution. If there is a point on the curve where $\kappa(s) = 0$ (and hence $N(s)$, $B(s)$ and $\tau(s)$ are undefined), the curve can be "twisted" around this point without changing the curvature or torsion at any other point. The *Mathematica* notebook *SpaceCurve.nb*, on the course website, contains a function that uses the Frenet-Serret equations to draw a curve in space, given the curvature and torsion functions. Experiment with it to see the effects of changing the curvature and torsion of a curve.

Exercise 15. *Show that if $\alpha(s)$ is a unit-speed curve, $\kappa(s) > 0$ is constant, and $\tau(s) = 0$, then $\alpha(s)$ is part of a circle of radius $1/\kappa$.*

It is all well and good to define curvature and torsion for unit-speed curves, but what if your curve is *not* unit speed? Parametrizing by arc length may be very difficult, or even impossible. The next exercise provides formulae for computing curvature and torsion for non-unit speed curves.

Exercise 16. *Show that if $\alpha : (a, b) \to \mathbb{R}^3$ is a regular curve (not necessarily unit speed), then the curvature and torsion are given by:*

$$\kappa(t) = \frac{\|\alpha'(t) \times \alpha''(t)\|}{\|\alpha'(t)\|^3} \quad \text{and} \quad \tau(t) = \frac{(\alpha'(t) \times \alpha''(t)) \cdot \alpha'''(t)}{\|\alpha'(t) \times \alpha''(t)\|^2}$$

[Hint: Let $\beta(s) = \alpha(t(s))$ be the arc-length parametrization of α, where $t(s)$ is the inverse of the arc-length function $s(t)$. Find expressions for $t'(s)$ and $t''(s)$, and compute the curvature $\kappa(t)$ using the chain rule. Then use the Frenet-Serret Theorem to compute the torsion.]

Exercise 17. *Determine the analog of the Frenet-Serret equations for a regular curve that is* not *unit speed.*

Exercise 18. *Let the curve C be the intersection of the cylinder $\{(x, y, z)|x^2 + y^2 = 1\}$ and the paraboloid $\{(x, y, z)|(x - 1)^2 + y^2 = z\}$.*

1. *Find a parametrization for C and compute its curvature and torsion.*
2. *Show that C is a planar curve.*

4 A Bit of Advanced Calculus

Just as curves are defined using functions from $\mathbb{R}$ to $\mathbb{R}^3$, surfaces are defined using functions from $\mathbb{R}^2$ to $\mathbb{R}^3$. We need to review how to differentiate these functions, and a couple of important theorems. We will begin with greater generality, looking at a function $f : \mathbb{R}^n \to \mathbb{R}^m$ for any n and m.

Definition 7. *Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be a function. We can break f into component functions, so:*

$$f(x_1, \ldots, x_n) = (f_1(x_1, \ldots, x_n), \ldots, f_m(x_1, \ldots, x_n))$$

*where $f_i : \mathbb{R}^n \to \mathbb{R}$. Then the **derivative of f at a point** $p = (p_1, \ldots, p_n)$ (also called the **Jacobian** of f at p) is the $m \times n$ matrix*

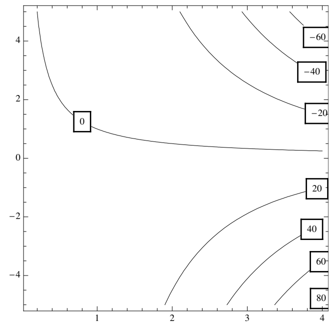
$$Df_p = \left(\frac{\partial f_i}{\partial x_j}\right) = \begin{pmatrix} \left.\frac{\partial f_1}{\partial x_1}\right|_p & \cdots & \left.\frac{\partial f_1}{\partial x_n}\right|_p \\ \vdots & \vdots & \vdots \\ \left.\frac{\partial f_m}{\partial x_1}\right|_p & \cdots & \left.\frac{\partial f_m}{\partial x_n}\right|_p \end{pmatrix}$$

Inverse Function Theorem. *Let $F : (U \subset \mathbb{R}^n) \to \mathbb{R}^n$ be a differentiable mapping and suppose, for some $u \in U$, $DF_u = \left(\frac{\partial F_i}{\partial x_j}\right) : \mathbb{R}^n \to \mathbb{R}^n$ is a linear isomorphism (i.e. $\det DF_u \neq 0$). Then there exists an open set $V \subset U$, containing u, and an open set $W \subset \mathbb{R}^n$, containing $F(u)$, such that $F : V \to W$ has a differentiable inverse $F^{-1} : W \to V$.*

The Inverse Function Theorem generalizes the familiar theorem from single-variable calculus that if $f : \mathbb{R} \to \mathbb{R}$ is a function with $f'(a) \neq 0$, then f is monotonic on some neighbourhood of a, so f is invertible near a.

Implicit Function Theorem. *Suppose $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^m$ is differentiable around $(u, v) \in \mathbb{R}^n \times \mathbb{R}^m$, and $f(u, v) = \mathbf{0}$. Let M be the $m \times m$ matrix $\left.\left(\frac{\partial f_i}{\partial x_{n+j}}\right)\right|_{(u,v)}$, where $1 \leq i, j \leq m$. If $\det M \neq 0$, then there exists an open set $U \subset \mathbb{R}^n$ containing u and an open set $V \subset \mathbb{R}^m$ containing v such that for every $r \in U$ there is a unique $s \in V$ such that $f(r, s) = \mathbf{0}$. If we define $g : U \to V$ by $g(r) = s$, then g is differentiable.*

For example, consider the function $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ defined by $f(x, y) = x - x^2y$. Observe that $f(1, 1) = 0$. In this case, $M = \left.\frac{\partial f}{\partial y}\right|_{(1,1)} = -x^2|_{(1,1)} = -1 \neq 0$. So the Implicit Function Theorem tells us that for values of x near 1, we can find a differentiable function $y(x)$ such that $f(x, y(x)) = 0$. In this case, it is easy to see that $y(x) = 1/x$. This is the level curve in the contour plot of $f(x, y)$ corresponding to the value $f(x, y) = 0$, as shown below. So the Implicit Function Theorem can be viewed as saying that "level spaces" (whether curves, surfaces or higher-dimensional spaces) are themselves graphs of differentiable functions.



5 Surfaces in Space

While a curve is usually defined by a single function $\alpha : (a, b) \to \mathbb{R}^3$, a surface is more complicated. There is no smooth way to map an open disk in $\mathbb{R}^2$ to cover, say, a sphere or a torus. Instead, we have to parametrize the surface in *patches*, which forces us to think about how these patches fit together. You can imagine that we are covering our surface with a series of local maps, like a Thomas Guide.

Definition 8. *Let S be a subset of $\mathbb{R}^3$. S is a **regular surface** if, for each point $p \in S$, there exists an open set $V \subset \mathbb{R}^3$, with $p \in V$, and a function $X : (U \subset \mathbb{R}^2) \to V \cap S$, where U is an open set in $\mathbb{R}^2$, satisfying the following conditions:*

1. *X is differentiable. I.e. if $X(u, v) = (f_1(u, v), f_2(u, v), f_3(u, v))$, then all partials of f_i of all orders exist.*
2. *X is a **homeomorphism**. This means that $X : U \to V \cap S$ is one-to-one and onto, and X has a* continuous *inverse $X^{-1} : V \cap S \to U$.*
3. *(Regularity)* $DX|_{(u,v)} = \begin{pmatrix} \frac{\partial f_1}{\partial u}(u,v) & \frac{\partial f_1}{\partial v}(u,v) \\ \frac{\partial f_2}{\partial u}(u,v) & \frac{\partial f_2}{\partial v}(u,v) \\ \frac{\partial f_3}{\partial u}(u,v) & \frac{\partial f_3}{\partial v}(u,v) \end{pmatrix}$ *has rank 2. This means the linear map $DX|_{(u,v)} : \mathbb{R}^2 \to \mathbb{R}^3$ is one-to-one.*

*X is called a **coordinate chart**, or a **patch**. A collection of coordinate charts which covers the surface is called an **atlas** for the surface.*

Exercise 19. *Perhaps the most familiar (non-planar) surface is the unit sphere $S = \{(x, y, z)|x^2 + y^2 + z^2 = 1\}$. In this exercise, you will look at two ways of covering the sphere with coordinate charts.*

1. *The most obvious way is to look at hemispherical charts. For example, let $U = \{(u, v)|u^2 + v^2 < 1\}$ be the* open *unit disk in $\mathbb{R}^2$, and let $X_1(u, v) = (u, v, \sqrt{1 - u^2 - v^2})$. Show that X_1 is a coordinate chart for the unit sphere.*
2. *How many such hemispherical charts do you need to cover the sphere? Find the charts. Be careful - don't forget the equator!*
3. *Another approach is to use spherical coordinates. Show how to cover the sphere with two coordinate charts using spherical coordinates, and verify that your proposed functions are coordinate charts.*

Exercise 20. *An important class of regular surfaces is the collection of graphs of smooth functions. If $f : \mathbb{R}^2 \to \mathbb{R}$ is a smooth function, its graph is the set $S = \{(u, v, f(u, v))|(u, v) \in \mathbb{R}^2\}$. Show that this graph is a regular surface which can be covered with a single coordinate chart.*

Exercise 21. *In Exercise 20, it is important that the function be smooth. For example, consider the cone $C = \{(x, y, z)|x^2 + y^2 = z^2,\ z \geq 0\}$.*

1. *Find a function whose graph is C.*
2. *Show that the coordinate chart from Exercise 20 is* not *a coordinate chart for the cone. Which point of the surface causes the problem?*

In fact, there is no *way to cover the cone with smooth coordinate charts. If we allow the charts to be only continuous, rather than smooth, then it is possible - the cone is a* topological *surface, but not a* differentiable *surface.*

Exercise 22. *Consider the set* $S = \{(x, y, z)| -1 < x < 1,\ y = 0,\ -1 < z < 1\} \cup \{(x, y, z)|x = 0,\ -1 < y < 1,\ -1 < z < 1\}$.

1. *Sketch a picture of* S.
2. *Is* S *a regular surface? Either cover* S *with coordinate charts, or explain why it is impossible.*

Finding explicit coordinate patches for a surface can be tedious. Sometimes, we need the charts - but other times, it's enough to know that the surface is regular without knowing the charts explicitly. Fortunately, this is often possible.

Definition 9. *Given a smooth map* $f : \mathbb{R}^n \to \mathbb{R}^m$*, with* $n > m$*, a point* $u \in \mathbb{R}^n$ *is a* ***critical point*** *of* f *if the linear map* $Df_u : \mathbb{R}^n \to \mathbb{R}^m$ *is not onto.* $f(u)$ *is called a* ***critical value*** *of* f*. If* $v \in \mathbb{R}^m$ *is not a critical value of* f*, then it is called a* ***regular value*** *of* f*.*

So a vector in $\mathbb{R}^m$ is a critical value if *any* of its preimages are critical points. It is a regular value if *none* of the preimages are critical points. In fact, the set of preimages of a regular value are particularly nice.

Theorem 2. *If* $f : (U \subset \mathbb{R}^3) \to \mathbb{R}$ *is a smooth function and* $r \in f(U)$ *is a regular value, then* $f^{-1}(r) = \{u \in U|f(u) = r\}$ *is a regular surface in* $\mathbb{R}^3$*.*

Proof. Let $p = (x_0, y_0, z_0) \in f^{-1}(r)$. Since r is a regular value, $Df_p = \left(\frac{\partial f}{\partial x}\Big|_p, \frac{\partial f}{\partial y}\Big|_p, \frac{\partial f}{\partial z}\Big|_p \right) \neq (0, 0, 0)$. Without loss of generality, assume $\frac{\partial f}{\partial z}\Big|_p \neq 0$.

Define $F : U \to \mathbb{R}^3$ by $F(x, y, z) = (x, y, f(x, y, z))$. Then the derivative of F at p is:

$$DF_p = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \frac{\partial f}{\partial x}\Big|_p & \frac{\partial f}{\partial y}\Big|_p & \frac{\partial f}{\partial z}\Big|_p \end{pmatrix} \implies \det(DF_p) = \frac{\partial f}{\partial z}\Big|_p \neq 0.$$

So we can apply the Inverse Function Theorem to F to "solve for z". There exists an open set $V \subset U$, containing p, and an open set $W \subset \mathbb{R}^3$, containing $F(p) = (x_0, y_0, r)$, such that $F : V \to W$ has a differentiable inverse $F^{-1} : W \to V$. Now our desired coordinate chart around p is defined on the open set $U' = \{(x, y) \in \mathbb{R}^2|(x, y, r) \in W\}$ (essentially, the projection to the xy-plane of the intersection of W with the plane $z = r$). The chart is defined by $X : U' \to V \cap f^{-1}(r)$, where $X(x, y) = F^{-1}(x, y, r)$. Since F^{-1} is smooth, and its derivative is a linear isomorphism, it is easy to check that X is a coordinate chart. Since we can construct such a chart around every point of $f^{-1}(r)$, the preimage is a regular surface. □

Exercise 23. *Use Theorem 2 to show that each of the following are regular surfaces. You do* not *need to construct an atlas of coordinate charts.*

1. *The sphere* $x^2 + y^2 + z^2 = 1$.
2. *The ellipsoid* $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$.
3. *The torus* $z^2 + \left(\sqrt{x^2 + y^2} - a\right)^2 = r^2$*, where* $0 < r < a$.

Notice that Theorem 2 is not an "if and only if" result. The next exercise shows why.

Exercise 24. *Let* $f(x, y, z) = (x + y + z + 1)^2$.

1. *Find the critical points and critical values of* f.
2. *For what values of* c *is the set* $f^{-1}(c)$ *a regular surface? (Check the critical values as well!)*
3. *Repeat (a) and (b) for* $f(x, y, z) = xyz^2$.

6 Functions on Surfaces

We want to be able to do calculus on surfaces. This means understanding what is means for a function on a surface, or a function from one surface to another, to be differentiable, and how to compute the derivative. We will begin by considering a real-valued function on a surface.

Given a regular surface S, a point $p \in S$ and a function $f : S \to \mathbb{R}$, what does it mean for f to be differentiable at p? Since S is a regular surface, there is a coordinate chart $X : U \to S$, where $U \subset \mathbb{R}^2$, such that $p \in X(U)$. We know what it means for the composition $f \circ X : (U \subset \mathbb{R}^2) \to \mathbb{R}$ to be differentiable; so we will say that f *is differentiable at* p if $f \circ X$ is differentiable at $X^{-1}(p)$.

But is this a well-defined notion? What if we used a *different* coordinate chart, say Y, around p? If $f \circ X$ is differentiable at $X^{-1}(p)$, do we know that $f \circ Y$ is differentiable at $Y^{-1}(p)$, and vice versa? The next theorem tells us that we do.

Theorem 3. *Suppose S is a regular surface and $X : (U \subset \mathbb{R}^2) \to \mathbb{R}^3$ and $Y : (V \subset \mathbb{R}^2) \to \mathbb{R}^3$ are coordinate charts with $W = X(U) \cap Y(V) \neq \emptyset$. Then the* ***change of coordinates function***

$$h = Y^{-1} \circ X : X^{-1}(W) \to Y^{-1}(W)$$

is differentiable and bijective (one-to-one and onto), and its inverse $h^{-1} = X^{-1} \circ Y$ is also differentiable.

Proof. We know that X is differentiable, so the trick is to show that Y^{-1} is differentiable. But we don't really know what this means yet. To deal with this, we use a trick to extend Y to a function from $\mathbb{R}^3$ to $\mathbb{R}^3$, and then use the Inverse Functiion Theorem to show the inverse is differentiable. See [1] for the details. □

So now we know that if $f \circ X$ is differentiable, so is $(f \circ X) \circ (X^{-1} \circ Y) = f \circ Y$, so our notion of a differentiable function on a surface is well-defined. We can now say this formally.

Definition 10. *If $f : S \to \mathbb{R}$ is a function and $p \in S$, then f is* ***differentiable at*** *p if for any coordinate chart $X : (U \subset \mathbb{R}^2) \to S$ with $p \in X(U)$, $f \circ X$ is differentiable at $X^{-1}(p)$ (it suffices to check one such chart). f is* ***differentiable on*** *S if it is differentiable at every point of S.*

We can extend these ideas to functions between two surfaces.

Definition 11. *Given a function $\phi : S_1 \to S_2$ between regular surfaces S_1 and S_2, and a point $p \in S_1$, we say that ϕ is* ***differentiable at*** *p if there exists coordinate charts $X : (U \subset \mathbb{R}^2) \to S_1$ and $Y : (V \subset \mathbb{R}^2) \to S_2$, with $p \in X(U)$ and $\phi(p) \in Y(V)$, such that $Y^{-1} \circ \phi \circ X : U \to V$ is differentiable at $X^{-1}(p)$.*

$$\begin{array}{ccc} S_1 & \xrightarrow{\phi} & S_2 \\ \uparrow{\scriptstyle X} & & \uparrow{\scriptstyle Y} \\ U & \xrightarrow{Y^{-1}\circ\phi\circ X} & V \end{array}$$

ϕ is ***differentiable*** *if it is differentiable at every point of S_1. ϕ is a* ***diffeomorphism*** *if it is differentiable, one-to-one and onto, and has a differentiable inverse. S_1 and S_2 are then said to be* ***diffeomorphic****.*

Exercise 25. *Show the following pairs of surfaces are diffeomorphic.*

1. *The paraboloid $z = x^2 + y^2$ and the plane $z = 0$.*
2. *The sphere $x^2 + y^2 + z^2 = 1$ and the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$.*

7 Tangent Planes

The best linear approximation to a curve $\alpha(t)$ at a point $\alpha(t_0)$ is the *tangent line* $\ell = \{\alpha(t_0) + r\alpha'(t_0) | r \in \mathbb{R}\}$. We use this linear approximation to compute the length of the curve, the angle at which two curves intersect, etc. Similarly, the best linear approximation to a surface S at a point p is the *tangent plane* to S at p. We will use these tangent planes to measure angle, length and area on surfaces, among other things. In this section, we will define tangent planes, and use them to define the derivative of a function between two surfaces.

We begin, as always, with a coordinate chart. Say that S is a regular surface, $p \in S$, and $X : U \to S$ is a coordinate chart with $X(u_0, v_0) = p$. X has three component functions, i.e. $X(u, v) = (x(u, v), y(u, v), z(u, v))$. We will use X to define two curves on S: $\alpha(v) = X(u_0, v)$ and $\beta(u) = X(u, v_0)$. These two curves intersect at $\alpha(v_0) = \beta(u_0) = p$. (Essentially, α and β are the images under X of the vertical and horizontal lines through (u_0, v_0) in U.)

The tangent vectors to these two curves at p are

$$\alpha'(v_0) = \frac{\partial}{\partial v} X(u_0, v)\bigg|_{v_0} = X_v(u_0, v_0) = DX_{(u_0,v_0)} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \partial x/\partial v \\ \partial y/\partial v \\ \partial z/\partial v \end{pmatrix}\Bigg|_{(u_0,v_0)}$$

$$\beta'(u_0) = \frac{\partial}{\partial u} X(u, v_0)\bigg|_{u_0} = X_u(u_0, v_0) = DX_{(u_0,v_0)} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \partial x/\partial u \\ \partial y/\partial u \\ \partial z/\partial u \end{pmatrix}\Bigg|_{(u_0,v_0)}$$

We will refer to these tangent vectors as X_v and X_u, respectively. The vectors are the columns of $DX_{(u_0,v_0)}$, which has rank 2, so X_u and X_v are linearly independent, and therefore span a plane.

Definition 12. *The **tangent plane to** S **at** p is $T_p(S) = span\{X_u(u_0, v_0), X_v(u_0, v_0)\}$. Note that this is a plane through the origin - the actual plane containing p is $p + T_p(S)$.*

Since X_u and X_v are independent, $X_u \times X_v \neq \mathbf{0}$, and is perpendicular to $T_p(S)$. We define

$$N_X(p) = \frac{X_u \times X_v}{\|X_u \times X_v\|} = \text{unit normal to } S \text{ at } p$$

Once again, we need to ask to what extent the tangent plane and the unit normal depend on the choice of coordinate chart. This is answered by the following theorem.

Theorem 4. *Let S be a regular surface, $p \in S$, and X and Y coordinate charts for S around p. So $p = X(u_0, v_0) = Y(\overline{u_0}, \overline{v_0})$. Then*

$$N_X(p) = \frac{X_u \times X_v}{\|X_u \times X_v\|}\bigg|_{(u_0,v_0)} = \pm \frac{Y_{\overline{u}} \times Y_{\overline{v}}}{\|Y_{\overline{u}} \times Y_{\overline{v}}\|}\bigg|_{(\overline{u_0},\overline{v_0})} = \pm N_Y(p)$$

where the sign is

$$\frac{\det(D(Y^{-1} \circ X))}{|\det(D(Y^{-1} \circ X))|}$$

(Recall that $Y^{-1} \circ X$ is differentiable by Theorem 3.)

Exercise 26. *Prove Theorem 4. [Hint: chain rule.]*

So while the unit normals determined by different coordinate charts may not be the same, they will be parallel, which means the orthogonal plane through the origin they determine will be the same. Therefore, the tangent plane $T_p(S)$ *is* well-defined.

Exercise 27. *Find an equation for the tangent plane to the ellipsoid $\frac{x^2}{4} + \frac{y^2}{16} + z^2 = 1$ at the point $\left(1, 2, \frac{1}{\sqrt{2}}\right)$.*

Exercise 28. *Say that 0 is a regular value of the differentiable function $f : \mathbb{R}^3 \to \mathbb{R}$, and let S be the regular surface $f^{-1}(0)$. Find an equation for $T_p(S)$, where $p = (x_0, y_0, z_0)$. Using your result, find the equation for the tangent plane to the hyperboloid $x^2 + y^2 - z^2 = 1$ at $p = (1, 2, 2)$.*

Exercise 29. *Let S^2 denote the unit sphere $\{(x, y, z)|x^2 + y^2 + z^2 = 1\}$. For any $p \in S^2$, show that the tangent plane $T_p(S^2)$ is perpendicular to the vector p.*

Sometimes we can find an atlas of coordinate charts so that whenever two charts overlap, the unit normal vectors are the same.

Definition 13. *A regular surface S is **orientable** if there is an atlas of charts $\{X_\alpha : U_\alpha \to S\}$ that cover S such that if $p \in X_\alpha(U_\alpha) \cap X_\beta(U_\beta)$, then $D(X_\beta^{-1} \circ X_\alpha)$ has positive determinant at $X_\alpha^{-1}(p)$ (and so $N_\alpha(p) = N_\beta(p)$).*

Orientable surfaces allow us to make a continuous, well-defined choice of unit normal vector at every point of the surface. Such a choice is called an *orientation* of the surface. In fact, an orientable surface has two possible orientations. For example, on a sphere one can choose the outward-pointing normal vector at every point, or the inward-pointing normal vector.

Exercise 30. *The most famous non-orientable surface is the* Möbius band.

1. *Use* Mathematica *to draw the surface defined by the coordinate charts* $X, \overline{X} : (0, 2\pi) \times (-1, 1) \to \mathbb{R}^3$*, where:*

$$X(u, v) = \left(\left(2 - v \sin \frac{u}{2}\right) \sin u, \left(2 - v \sin \frac{u}{2}\right) \cos u, v \cos \frac{u}{2}\right)$$

$$\overline{X}(\overline{u}, \overline{v}) = \left(\left(2 - \overline{v} \sin \left(\frac{\pi}{4} + \frac{\overline{u}}{2}\right)\right) \cos \overline{u}, \left(2 - \overline{v} \sin \left(\frac{\pi}{4} + \frac{\overline{u}}{2}\right)\right) \sin \overline{u}, \overline{v} \cos \left(\frac{\pi}{4} + \frac{\overline{u}}{2}\right)\right)$$

2. *Find the change of coordinates where the charts overlap, and show that the surface is not orientable.*

We know that the tangent plane $T_p(S)$ contains (by definition) the tangent vectors to two curves in S passing through p. In fact, it contains the tangent vector to *every* curve in S passing through p; moreover, every vector in the tangent plane is the tangent vector to some curve in S through p.

Theorem 5. *A vector* $\mathbf{v} \in \mathbb{R}^3$ *is in* $T_p(S)$ *if and only if there exists a differentiable curve* $\alpha : (-\epsilon, \epsilon) \to S$ *with* $\alpha(0) = p$ *and* $\alpha'(0) = \mathbf{v}$.

Exercise 31. *Prove Theorem 5. [Hint: for the "only if" part, write* $\mathbf{v}$ *as a linear combination of* X_u *and* X_v*, and find a curve with this tangent vector. For the "if" part, consider the curve* $X^{-1} \circ \alpha$*, and write* α *as the composition of this curve with* X*.]*

Now that we've defined the tangent plane, we can use it to define the derivative of functions between surfaces. Recall that when we have a function $f : \mathbb{R}^n \to \mathbb{R}^m$, the derivative Df_p at a point $p \in \mathbb{R}^n$ is a *linear* map from $\mathbb{R}^n$ to $\mathbb{R}^m$. When we have a function between two surfaces, the derivative is again a linear map (calculus is all about making the non-linear linear!), this time between the tangent planes to the two surfaces. To define this linear map, we will use the fact that every vector in the tangent plane is the tangent vector to a curve in the surface.

Definition 14. *Let* S_1 *and* S_2 *be regular surfaces, and let* $\phi : S_1 \to S_2$ *be a function which is differentiable at* $p \in S_1$*. Then* $D\phi_p : T_p(S_1) \to T_{\phi(p)}S_2$ *is defined by*

$$D\phi_p(\alpha'(0)) = \frac{d}{dt}(\phi \circ \alpha)|_{t=0}$$

where $\alpha(t)$ *is a smooth curve in* S_1 *with* $\alpha(0) = p$.

Since, by Theorem 5, every vector in $T_p(S_1)$ is the tangent vector to some curve through p, this defines $D\phi_p$ on all of $T_p(S_1)$.

Exercise 32. *Show that* $D\phi_p$ *is a linear transformation.*

Exercise 33. *Prove the chain rule for functions between surfaces. In other words, if* $\phi : S_1 \to S_2$ *and* $\psi : S_2 \to S_3$ *are differentiable functions, show that:*

$$D(\psi \circ \phi)_p = D\psi_{\phi(p)} \circ D\phi_p$$

8 The First Fundamental Form

In this section we will see how to use the tangent plane to compute lengths, angles and areas on a surface in $\mathbb{R}^3$. First, let's remember how we can use the dot product to compute lengths and angles in $\mathbb{R}^2$.

$$v \in \mathbb{R}^2 \implies \|v\|^2 = v \cdot v$$

$$v, w \in \mathbb{R}^2 \implies v \cdot w = \|v\|\|w\| \cos \theta \implies \theta = \arccos \left(\frac{v \cdot w}{\|v\|\|w\|}\right)$$

where θ is the angle between the vectors v and w.

Since the tangent plane $T_p(S)$ is the best linear approximation to S near p, the idea is to use the dot product in $T_p(S)$ to compute lengths and angles in S near p.

Definition 15. *If* S *is a regular surface, and* $p \in S$*, the* ***first fundamental form*** $I_p : T_p(S) \times T_p(S) \to \mathbb{R}$ *is the inner product on* $T_p(S)$ *induced by the dot product on* $\mathbb{R}^3$*. So* $I_p(v, w) = v \cdot w = \langle v, w \rangle$.

To actually compute the first fundamental form, we need to express it in terms of whatever coordinate chart X we are using around p. If $p \in X(U)$, then $T_p(S)$ has a basis $\{X_u, X_v\}$ (evaluated at $X^{-1}(p)$). So if $v, w \in T_p(S)$, we can write $v = aX_u + bX_v$ and $w = cX_u + dX_v$. Then

$$I_p(v,w) = \langle aX_u + bX_v, cX_u + dX_v \rangle = ac\langle X_u, X_u\rangle + (ad+bc)\langle X_u, X_v\rangle + bd\langle X_v, X_v\rangle$$

This leads us to define the *component functions of the metric*

$$E(u,v) = \langle X_u, X_u\rangle$$
$$F(u,v) = \langle X_u, X_v\rangle$$
$$G(u,v) = \langle X_v, X_v\rangle$$

Then we can write the first fundamental form using matrix multiplication as

$$I_p(v,w) = acE + (ad+bc)F + bdG = \begin{pmatrix} a & b \end{pmatrix} \begin{pmatrix} E & F \\ F & G \end{pmatrix} \begin{pmatrix} c \\ d \end{pmatrix} = v^t \begin{pmatrix} E & F \\ F & G \end{pmatrix} w$$

Exercise 34. *Compute the component functions E, F and G of the first fundamental form for the following coordinate charts:*

1. $X(u,v) = (u,v,0),\ u,v \in \mathbb{R}$ *(the xy-plane)*
2. $X(u,v) = (u,v,f(u,v)),\ u,v \in \mathbb{R}$ *(the graph of the function $f(x,y) = z$)*
3. $X(u,v) = (\cos u, \sin u, v),\ 0 < u < 2\pi,\ v \in \mathbb{R}$ *(the cylinder $x^2 + y^2 = 1$)*
4. $X(u,v) = (\cos u \sin v, \sin u \sin v, \cos v),\ -\pi < u < \pi,\ 0 < v < \pi$ *(the unit sphere, in spherical coordinates)*

What do you notice about the component functions for the plane and the cylinder? How do you explain this?

We can use the first fundamental form to compute the length of a curve on a surface. Of course, we already know how to compute the length of a curve in $\mathbb{R}^3$; here, we are interested in the case where the curve on S is the image of a known curve in $\mathbb{R}^2$ under some coordinate chart. In other words, we have a curve $\beta : (a,b) \to (U \subset \mathbb{R}^2)$ and a coordinate chart $X : U \to S$, and we would like to know the length of the curve $\alpha = X \circ \beta$. We could, of course, compute the composition explicitly and use our previous techniques. But if we are going to have to do this multiple times for a given surface, it is convenient to save ourselves some work by computing the first fundamental form once, and using it with the different curves β.

Let's say that $\beta(t) = (u(t), v(t))$, so $\alpha(t) = X(u(t), v(t))$. We want to know the length $s(t)$ of the curve α between $\alpha(t_0)$ and $\alpha(t)$. We know from our work on curves in space that

$$s(t) = \int_{t_0}^{t} \|\alpha'(r)\| dr = \int_{t_0}^{t} \sqrt{I_{\alpha(r)}(\alpha'(r), \alpha'(r))} dr$$

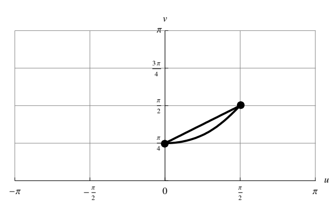
Since $\alpha(t) = X(u(t), v(t))$, we can apply the chain rule.

$$\alpha'(r) = \frac{d}{dr} X(u(r), v(r)) = X_u u'(r) + X_v v'(r) \implies I_{\alpha(r)}(\alpha'(r), \alpha'(r)) = E(u'(r))^2 + 2Fu'(r)v'(r) + G(v'(r))^2$$

$$\implies s(t) = \int_{t_0}^{t} \sqrt{E(u'(r))^2 + 2Fu'(r)v'(r) + G(v'(r))^2} dr$$

We will often write $ds = \sqrt{E(u'(r))^2 + 2Fu'(r)v'(r) + G(v'(r))^2} dr$, or $ds^2 = Edu^2 + 2Fdudv + Gdv^2$. ds is called the *line element on S* or the *element of arc length*.

Exercise 35. *Consider the unit sphere with the coordinate chart $X(u,v) = (\cos u \sin v, \sin u \sin v, \cos v)$, $-\pi < u < \pi$, $0 < v < \pi$. In the domain $U = (-\pi, \pi) \times (0, \pi)$, consider the curves $v = \frac{1}{2}u + \frac{\pi}{4}$ and $v = \arctan(\sec u)$ between the points $(0, \pi/4)$ and $(\pi/2, \pi/2)$, as shown below.*



1. *Which of these two curves is shorter in U? (You do not need to compute the lengths.)*
2. *Compute the lengths of the images of these curves on the sphere under X. Use* Mathematica *or some other computer algebra system to compute the definite integrals. Which curve is shorter on the sphere?*
3. *What does this mean when you are looking at a standard (flat) map of the world?*

We can also use the first fundamental form to compute angles on a surface, just as we normally do using the dot product. Given curves $\alpha(t)$ and $\beta(t)$ on a surface S such that $\alpha(0) = \beta(0) = p$, the angle between the curves at p is given by

$$\theta = \arccos\left(\frac{I_p(\alpha'(0), \beta'(0))}{\sqrt{I_p(\alpha'(0), \alpha'(0)) I_p(\beta'(0), \beta'(0))}}\right)$$

In particular, note that the angle between the basis vectors X_u and X_v of a tangent plane $T_p(S)$ is $\arccos\left(\frac{F}{\sqrt{EG}}\right)$. So X_u and X_v are perpendicular if and only if $F = 0$. If this occurs throughout the coordinate chart, then X is called an *orthogonal parametrization*. For example, the parametrization of the sphere using spherical coordinates is orthogonal, but the parametrization using hemispheres is not.

Exercise 36. *Assume X is a coordinate patch for S, and $\alpha(t) = X(a(t), b(t))$ and $\beta(t) = X(c(t), d(t))$. Further assume that $\alpha(0) = \beta(0) = p$.*

1. *Rewrite the formula for the angle between $\alpha(t)$ and $\beta(t)$ at p in terms of a, b, c, d and the component functions of the first fundamental form.*
2. *Compute the angle between the curves in Exercise 35 in U at the point $(0, \pi/4)$.*
3. *Compute the angle between the images of the curves in Exercise 35 on the unit sphere at the point $(1/\sqrt{2}, 0, 1/\sqrt{2})$.*
4. *Does this coordinate chart preserve angles?*

Finally, we can also use the first fundamental form to compute area. Given a surface S and a coordinate chart $X : U \to S$, we want to compute the area of a region $R \subset X(U)$ on the surface by doing some kind of surface integral. Just as in Multivariable Calculus, we do this surface integral by pulling it back to a more familiar double integral over a region in the plane, by integrating over $X^{-1}(R) \subset U$.

Recall that the area of the parallelogram in the plane determined by the vectors v and w is $\|v \times w\|$. In U, the area element is the rectangle with sides du and dv, and area $dudv$. Under the coordinate chart X, the image of du is $X_u du$, and the image of dv is $X_v dv$. The area of the infinitesimal parallelogram bounded by these two vectors is $\|X_u \times X_v\| dudv$. So

$$\text{area of } R = \iint_R dA = \iint_{X^{-1}(R)} \|X_u \times X_v\| dudv$$

Since $\|X_u \times X_v\|^2 = \|X_u\|^2 \|X_v\|^2 - \langle X_u, X_v \rangle^2 = EG - F^2$, we can write this as

$$\text{area of } R = \iint_{X^{-1}(R)} \sqrt{EG - F^2} dudv$$

Exercise 37. *Consider the surface T with coordinate chart $X : (0, 2\pi) \times (0, 2\pi) \to \mathbb{R}^3$ defined by $X(u, v) = ((a + r\cos u)\cos v, (a + r\cos u)\sin v, r\sin u)$, where a, r are constants with $r < a$.*

1. *What is this surface?*
2. *Find the component functions of the first fundamental form for this chart.*
3. *Compute the area of the surface. Why can you compute the area of the entire surface from this single chart?*

9 The Gauss Map

Now that we've defined surfaces and discussed how to measure lengths, angles and areas on surfaces, we want to measure the *curvature* of a surface. For curves, the Frenet-Serrat Theorem shows that curvature and torsion together are used to measure changes in the unit normal vector. For surfaces, we will generalize this idea by measuring changes in the normal vector to a surface S.

Definition 16. *Let S be an orientable surface, with a selected orientation (continuous choice of normal vector at each point). Define the **Gauss map** to be the mapping $N : S \to S^2$, the unit sphere in $\mathbb{R}^2$ centered at the origin, given by associating to each point in S the unit normal at that point.*

Locally, if we have a coordinate chart $X : U \to S$, we can write the Gauss map as

$$N(p) = \frac{X_u \times X_v}{\|X_u \times X_v\|}$$

The Gauss map does depend on our coordinate chart - switching the coordinates u and v will reverse the direction of the normal vector (and the orientation of the surface). The restriction that S be orientable ensures that charts can be chosen so that the Gauss map is both continuous and well-defined.

Exercise 38. *Show that, for a regular surface, the Gauss map is differentiable.*

Since the Gauss map is differentiable, it has a differential map $DN_p : T_p(S) \to T_{N(p)}(S^2)$. Recall from Exercise 29 that $T_{N(p)}(S^2)$ is perpendicular to $N(p)$, as is $T_p(S)$. So $T_{N(p)}(S^2) = T_p(S)$, and we can view DN_p as a linear mapping $DN_p : T_p(S) \to T_p(S)$.

Exercise 39. *For each of the following surfaces, find a coordinate chart X for the surface and compute the Gauss map $N(p)$ in terms of X (i.e. compute $N \circ X : U \to S^2$).*

1. *The cylinder $x^2 + y^2 = r^2$.*
2. *The sphere $x^2 + y^2 + z^2 = r^2$.*
3. *The saddle $z = x^2 - y^2$.*

10 The Second Fundamental Form

The linear mapping DN_p has a special property with respect to the first fundamental form.

Definition 17. *Let V be a real vector space with an inner product $\langle , \rangle : V \times V \to \mathbb{R}$. If $A : V \to V$ is a linear mapping, then A is said to be **self-adjoint** with respect to this inner product if, for all $v, w \in V$,*

$$\langle A(v), w\rangle = \langle v, A(w)\rangle$$

Theorem 6. *The differential $DN_p : T_p(S) \to T_p(S)$ is self adjoint with respect to the first fundamental form.*

Proof. We need to show that $I_p(DN_p(v), w) = I_p(v, DN_p(w))$ for any $v, w \in T_p(S)$. Since I_p is linear in each variable, it suffices to prove this for the basis vectors X_u and X_v; i.e. to show $I_p(DN_p(X_u), X_v) = I_p(X_u, DN_p(X_v))$.

Recall, from the chain rule, that $DN_p(X_u) = \frac{\partial}{\partial u}(N \circ X) = N_u$ and $DN_p(X_v) = \frac{\partial}{\partial v}(N \circ X) = N_v$. Since the normal vector $N(p)$ is perpendicular to $T_p(S)$, we know that $I_p(N, X_u) = 0$ and $I_p(N, X_v) = 0$. Differentiating the first equation with respect to v and the second with respect to u gives

$$I_p(N_v, X_u) + I_p(N, X_{uv}) = 0 \qquad \text{and} \qquad I_p(N_u, X_v) + I_p(N, X_{vu}) = 0$$

Since coordinate patches are differentiable, $X_{uv} = X_{vu}$, so

$$I_p(N_u, X_v) = -I_p(N, X_{vu}) = -I_p(N, X_{uv}) = I_p(X_u, N_v)$$

□

Why do we care if the derivative of the Gauss map is self-adjoint? Because self-adjoint linear maps have many nice properties.

Exercise 40. *Suppose V is a 2-dimensional vector space with an inner product $\langle , \rangle$, and $A : V \to V$ is a self-adjoint linear map with respect to this inner product.*

1. *Show that the matrix for A is symmetric with respect to any basis for V. I.e. the matrix for A has the form*

$$A = \begin{pmatrix} a & b \\ b & d \end{pmatrix}$$

2. *Show that A has two real (not necessarily distinct) eigenvalues.*

3. *If A has two distinct eigenvalues, show that the corresponding eigenvectors are perpendicular.*

4. *Assume A has distinct real eigenvalues $\lambda_1 > \lambda_2$ and corresponding unit-length eigenvectors e_1 and e_2. Suppose $v \in V$ with $\|v\| = 1$. Then show that we can write $v = \cos(\theta)e_1 + \sin(\theta)e_2$ for some θ, and $\langle A(v), v\rangle = \lambda_1 \cos^2\theta + \lambda_2 \sin^2\theta$. Furthermore, show that the expression $\langle A(v), v\rangle$ has a maximum value of λ_1 when $\theta = 0$ or π and a minimum value of λ_2 when $\theta = \pm\theta/2$.*

Since the map DN_p is self-adjoint, the last part of Exercise 40 makes it natural to consider $I_p(DN_p(v), v)$.

Definition 18. *At a point p in a surface S the* ***second fundamental form*** *on S at p is the quadratic form $II_p : T_p(S) \to \mathbb{R}$ defined by*

$$II_p(v) = -I_p(DN_p(v), v)$$

If v is a unit vector, the second fundamental form has a very nice geometric interpretation in terms of the curvature of curves on the surface S passing through the point p. Let v be a unit tangent vector in $T_p(S)$. Consider the plane P in $\mathbb{R}^3$ with basis $\{v, N(p)\}$, translated to pass through p. The intersection of P and S is a planar curve in S, passing through p.

Suppose $c_v(s)$ is a unit-speed parametrization of this curve such that $c_v(0) = p$ and $c_v'(0) = v$. Then $c_v''(0)$ is a vector in P such that $c_v''(0) \perp c_v'(0) = v$ (since $c_v(s)$ is unit speed). Also recall that $\|c_v''(s)\| = \kappa(s)$, the curvature of c_v. So $c_v''(0) = \pm\kappa(0)N(p)$.

Definition 19. *The* ***normal curvature of*** *S* ***in the*** *v* ***direction at*** *p, denoted $k_n(v)$, is defined by*

$$k_n(v) = I_p(c_v''(0), N(p)) = \pm\kappa(0)$$

Theorem 7. *For $v \in T_p(S)$ with $\|v\| = 1$, $II_p(v) = k_n(v)$.*

Exercise 41. *Prove Theorem 7. [Hint: observe that $I_{c_v(s)}(N(c_v(s)), c_v'(s)) = 0$.]*

Since DN_p is self-adjoint, the results of Exercise 40 combined with Theorem 7 give us the following theorem (originally proved by Euler in 1760).

Theorem 8. *If the normal curvature $k_n(v)$ is not a constant function of v, then there are precisely two unit tangent vectors X_1 and X_2 such that $k_n(X_1) = k_1$ is maximal and $k_n(X_2) = k_2$ is minimal. Furthermore, X_1 is perpendicular to X_2, and for any unit vector v there is an angle θ such that $k_n(v) = k_1\cos^2\theta + k_2\sin^2\theta$.*

k_1 and k_2 are called the *principal curvatures* of S at p and X_1 and X_2 are called the *principal directions.* The principal directions are the eigenvectors of DN_p; the principal curvatures are the negatives of the eigenvalues (the sign change is due to the negative sign in the definition of the second fundamental form). If we assume that coordinates of $\mathbb{R}^3$ are chosen so that p is at the origin and $T_p(S)$ is the xy-plane, then Theorem 8 implies that, near p, S can be described as the graph of a function

$$z = k_1x^2 + k_2y^2 + R(x, y), \qquad \text{where } \lim_{x,y\to 0} \frac{R(x, y)}{x^2 + y^2} = 0$$

So, near p, the surface is approximated by the quadratic surface $z = k_1x^2 + k_2y^2$. What this surface looks like depends on the values of k_1 and k_2.

CASE 1: $k_1, k_2 > 0$ or $k_1, k_2 < 0$ at p. These points are called *elliptic* and the quadratic surface is a paraboloid. Near p, the surface lies on one side of the tangent plane. For example, on spheres and ellipsoids, all points are elliptic.

CASE 2: $k_1 > 0$, $k_2 < 0$. These points are called *hyperbolic* and the quadratic surface is a hyperbolic paraboloid (i.e. a "saddle"). Near p, the surface has points on both sides of the tangent plane.

CASE 3: $k_1 > 0, k_2 = 0$ or $k_1 = 0$, $k_2 < 0$. These points are called *parabolic* and the quadratic surface is a parabolic cylinder $z = k_1x^2$ or $z = k_2y^2$. Near p, all points are on the same side of the tangent plane, or lie on the tangent plane. For example, all points of a cylinder are parabolic.

CASE 4: $k_1 = k_2 = 0$. These points are called *planar*. All points of a plane are (unsurprisingly) planar; but other functions such as $f(x, y) = x^3$ or $f(x, y) = \pm x^4$ have graphs with planar points (in these cases, when $x = 0$), and the local behavior may vary.

If k_n is constant, and nonzero, at p (so $k_1 = k_2 \neq 0$), then p is called an *umbilic* point. For example, all points on a sphere are umbilic.

11 Gaussian Curvature

If we write the derivative DN_p of the Gauss map as a matrix with respect to the orthonormal basis $\{X_1, X_2\}$ given by the principal directions at p, we get

$$DN_p = \begin{pmatrix} -k_1 & 0 \\ 0 & -k_2 \end{pmatrix}.$$

Definition 20. *The* ***Gaussian curvature*** *of S at p is defined by*

$$K(p) = \det(DN_p) = k_1 k_2$$

Observe that if p is an *elliptic* point then $K(p) > 0$, if p is *hyperbolic* then $K(p) < 0$, and if p is *parabolic* or *planar* then $K(p) = 0$. We should also note that the Gaussian curvature does *not* depend on the choice of coordinate chart (i.e. the choice of unit normal, or the choice of basis for $T_p(S)$). Reversing the unit normal reverses the signs of both k_1 and k_2, but the product $K(p)$ stays the same. Changing the basis for $T_p(S)$ will change the matrix for DN_p, but not its determinant (which is always the product of the eigenvalues).

Exercise 42. *You found the Gauss maps for the following surfaces in Exercise 39. Now find the matrix for DN_p with respect to the basis $\{X_u, X_v\}$. Next find the principal curvatures and directions (i.e. compute the eigenvalues and eigenvectors for the matrix for DN_p). Finally, compute the determinant of DN_p in each case to find the Gaussian curvature.*

1. *The cylinder $x^2 + y^2 = r^2$.*
2. *The sphere $x^2 + y^2 + z^2 = r^2$.*
3. *The saddle $z = x^2 - y^2$.*

Currently, computing the Gaussian curvature requires finding the matrix for DN_p with respect to some basis for $T_p(S)$, as in Exercise 42. However, computing the derivative of the Gauss map is often laborious (since it involves both a quotient and a cross product). We are going to develop another way of computing the Gaussian curvature which relies on computing higher derivatives of the coordinate chart X rather than the derivative of $N(p)$; this is usually easier.

Recall that, given a coordinate chart X around $p \in S$,

$$N(p) = \frac{X_u \times X_v}{\|X_u \times X_v\|}$$

$$N_u = DN_p(X_u) = a_{11} X_u + a_{21} X_v$$
$$N_v = DN_p(X_v) = a_{12} X_u + a_{22} X_v$$

So the matrix for DN_p is

$$DN_p = \begin{pmatrix} N_u & N_v \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

Since $N(p) \cdot X_u = 0$ and $N(p) \cdot X_v = 0$, the product rule gives us

$$\frac{\partial}{\partial u}(N \cdot X_u) = N_u \cdot X_u + N \cdot X_{uu} = 0 \implies N \cdot X_{uu} = -N_u \cdot X_u$$

$$\frac{\partial}{\partial v}(N \cdot X_u) = N_v \cdot X_u + N \cdot X_{uv} = 0 \implies N \cdot X_{uv} = -N_v \cdot X_u$$

$$\frac{\partial}{\partial u}(N \cdot X_v) = N_u \cdot X_v + N \cdot X_{vu} = 0 \implies N \cdot X_{vu} = -N_u \cdot X_v$$

$$\frac{\partial}{\partial v}(N \cdot X_v) = N_v \cdot X_v + N \cdot X_{vv} = 0 \implies N \cdot X_{vv} = -N_v \cdot X_v$$

We define the *coefficients of the second fundamental form* as follows

$$e = II_p(X_u) = -N_u \cdot X_u = N \cdot X_{uu}$$

$$f = -I_p(DN_p(X_v), X_u) = -N_v \cdot X_u = N \cdot X_{uv} = N \cdot X_{vu} = -N_u \cdot X_v$$

$$g = II_p(X_v) = -N_v \cdot X_v = N \cdot X_{vv}$$

Since $N_u = a_{11}X_u + a_{21}X_v$ and $N_v = a_{12}X_u + a_{22}X_v$, we find

$$e = -(a_{11}X_u + a_{21}X_v) \cdot X_u = -(a_{11}E + a_{21}F)$$

$$f = -(a_{12}X_u + a_{22}X_v) \cdot X_u = -(a_{12}E + a_{22}F)$$

$$f = -(a_{11}X_u + a_{21}X_v) \cdot X_v = -(a_{11}F + a_{21}G)$$

$$g = -(a_{12}X_u + a_{22}X_v) \cdot X_v = -(a_{12}F + a_{22}G)$$

Rewriting this system of equations as a matrix equation, we obtain

$$-\begin{pmatrix} e & f \\ f & g \end{pmatrix} = \begin{pmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{pmatrix} \begin{pmatrix} E & F \\ F & G \end{pmatrix}$$

Solving gives us

$$\begin{aligned} (DN_p)^t = \begin{pmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{pmatrix} &= -\begin{pmatrix} e & f \\ f & g \end{pmatrix} \begin{pmatrix} E & F \\ F & G \end{pmatrix}^{-1} \\ &= -\frac{1}{EG - F^2} \begin{pmatrix} e & f \\ f & g \end{pmatrix} \begin{pmatrix} G & -F \\ -F & E \end{pmatrix} \\ &= -\frac{1}{EG - F^2} \begin{pmatrix} eG - fF & fE - eF \\ fG - gF & gE - fF \end{pmatrix} \end{aligned}$$

So

$$K(p) = \det(DN_p) = \det((DN_p)^t) = \frac{eg - f^2}{EG - F^2}$$

Exercise 43. *Use this formula to compute the Gaussian curvature at each point of the following surfaces (compute e, f, g by computing $N \cdot X_{uu}$, $N \cdot X_{uv}$ and $N \cdot X_{vv}$, respectively). Compare to Exercise 42.*

1. *The paraboloid $z = x^2 + y^2$.*
2. *The sphere $x^2 + y^2 + z^2 = r^2$*

Exercise 44. *The* ***pseudosphere*** *is the surface of revolution of the tractrix. The tractrix is the curve followed by weight being dragged by a rope of fixed length whose other end is moving along a fixed line. A parametrization for the tractrix in the xy-plane (where the weight begins at $(1, 0)$, the rope is length 1, and the end is moved along the positive y-axis) is*

$$\alpha(t) = (\sin t, \ln\tan(t/2) + \cos t), \;\; \pi/2 \le t < \pi$$

The pseudosphere is the result of revolving this curve around the y-axis.

1. *Find a parametriazation for the pseudosphere.*
2. *Compute the Gaussian curvature for the pseudosphere. Why do you think it is called the pseudosphere?*

Exercise 45. *The* ***trace*** *of a square matrix is the sum of the elements along the diagonal. It is a fact that the trace of a diagonalizable matrix is the sum of the eigenvalues, with multiplicity. The* ***mean curvature*** *of a surface S at a point p is defined as $H(p) = -\frac{1}{2}\mathrm{tr}(DN_p) = \frac{1}{2}(k_1 + k_2)$.*

1. *Find a formula for the mean curvature $H(p)$ in terms of E, F, G, e, f, g. [Hint: find formulas for a_{ij}.]*
2. *Show that the principal curvatures k_1 and k_2 are given by $H \pm \sqrt{H^2 - K}$.*

Exercise 46. *Let S be the graph of a differentiable function $z = f(x, y)$. Find expressions for the Gaussian and mean curvatures of S.*

Exercise 47. *Compute the coefficients of the first fundamental form, the normal vector, the coefficients of the second fundamental form, and the Gaussian, mean and principal curvatures for each of the following surfaces.*

1. *The torus defined by* $X(u,v) = ((a + b\cos v)\cos u, (a + b\cos v)\sin u, b\sin v)$.
2. *The helicoid defined by* $X(u,v) = (v\cos u, v\sin u, u)$.
3. *Enneper's minimal surface* $X(u,v) = (u - u^3/3 + uv^2, v - v^3/3 + vu^2, u^2 - v^2)$.

Exercise 48. *Show that the sum of the normal curvatures for* any *pair of perpendicular directions at a point* $p \in S$ *is* $2H(p)$.

12 Where do we go from here?

We have only begun the study of differential geometry. In this final section, we will see some of the great results you will encounter if you continue to study the subject.

When we study a surface in $\mathbb{R}^3$, we don't really care if the surface is moved or rotated - it's still the same surface, just in a different position. So we are interested in properties of the surface which are preserved under *isometries* in $\mathbb{R}^3$. Properties which *are* preserved by isometries are called *intrinsic*; properties which are not preserved by isometries are called *extrinsic*. One of C.F. Gauss's most important results in differential geometry was his *Theorema Egregium* ("Remarkable Theorem").

Theorema Egregium. *Gaussian curvature is intrinsic.*

In fact, the coefficients E, F, G, e, f, g of the first and second fundamental forms completely determine a surface up to isometry (at least locally), just as curvature and torsion determine a curve in the Fundamental Theorem for Curves in Space.

In classical geometry, a line is the shortest distance between two points. In a more general surface, the shortest path between two points is more complicated. A *geodesic* is a curve which minimizes the distance between two points on the curve. So lines are geodesics in the plane, and great circles are geodesics on a sphere. Just as we study triangles and polygons in the plane, in other surfaces we study figures bounded by geodesic curves. Perhaps the most famous result about such figures is the *Gauss-Bonnet Theorem.*

Gauss-Bonnet Theorem. *If* R *is a simply connected region (i.e. no holes) in a regular surface* S *bounded by a piecewise geodesic curve* α *making exterior angles* $\theta_1, \theta_2, \ldots, \theta_n$ *at the vertices of* α*, then*

$$\iint_R K\, dA = 2\pi - \sum_{j=1}^{n} \theta_n$$

In particular, if S is a surface of constant curvature (such as the sphere or pseudosphere), then we can compute the area of a figure in terms of its external angles. On a plane, this theorem tells us that the sum of the external angles of a polygon is always 2π, and so the sum of the internal angles is π. On a sphere, the sum of the external angles is always less than 2π (so the sum of the internal angles is greater than π), and on a pseudosphere the sum of the external angles is always greater than 2π (so the sum of the internal angles is less than π). If $K \neq 0$, this means that two geodesic polygons that are similar (i.e. have the same angles) must have the same areas, and are actually congruent!

References

[1] J. McCleary, *Geometry from a Differentiable Viewpoint*, Cambridge University Press, 1994.